A Dynamic Network Economic Model of a Service-Oriented Internet with Price and Quality Competition

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Abstract In this paper, we develop a dynamic network economic model of a service-oriented Internet with price and quality competition using projected dynamical systems theory. This research, is motivated, in part, by the Future Generation Internet (FGI), which will need to handle a spectrum of requirements of next-generation applications. The decision-makers are the content providers and the transport network providers who select the prices that they charge as well as the quality levels of their content and network provision, respectively. Consumers, that is, users at the demand markets, respond through the demand functions which reflect the attractiveness of the composition of content and network services as reflected by the prices charged and the quality levels. We prove that the stationary points of the projected dynamical system coincide with the set of solutions of a variational inequality problem and

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provide qualitative analysis, including stability analysis results. In addition, we describe an algorithm, which yields a discrete-time approximation of the continuous-time adjustment processes until a stationary point, equivalently, an equilibrium, is achieved. The computational procedure is then applied to solve numerical examples in order to demonstrate the generality of the framework.

1 Introduction

The Internet has transformed the way in which we conduct business and perform economic and financial transactions, communicate and obtain information, and even acquire music and entertainment. Today, services, which were not even envisioned early in the Internet age, such as cloud computing and video streaming, are becoming mainstream. Nevertheless, providers are facing many challenges in determining technical and economic solutions in providing services (see Wolf et al. (2012)) with a notable challenge being how to price and bill these services. Equilibrium models for Internet networks generally assume basic economic relationships and consider price as the only factor that affects demand (cf. Laffont et al. (2003), Zhang et al. (2010), and Musacchio et al. (2011)). However, in new paradigms for the Internet, price is not the only factor, and Quality of Service (QoS), as the ability to provide different priorities to applications, users, or data flows, is rising to the fore, due, in part, to increasingly demanding consumers.

Since the demands on the current Internet are pushing the infrastructure beyond what it can provide, the future Internet will need to handle a plethora of requirements of next-generation applications and new users' requirements comprising mobility, security, and flexibility. The Future Generation Internet is expected to be service-oriented with each provider offering one or more specific services. In such an Internet with services of comparable functionalities, but varying quality levels, services are available at different costs in the service marketplace, so that users can decide which services from which service providers to select. As noted in Saberi, Nagurney, and Wolf (2013), the FGI is expected to include multi-tier service providers, such as content service providers and network service providers. A content service provider is a website or organization that handles the distribution of online content such as blogs, videos, music or files. A network service provider refers to a company that offers Internet access or, in general, is an entity that provides data transportation, which offer equal or rather similar services at different QoS levels and costs. In this paper, we use "Content Provider (CP)" instead of content service provider, and "Network Provider (NP)" in place of network service provider, for simplicity, and the fact that any provider offers a service, which can be either a content or a network service.

The economic complexity associated with designing the Future Generation Internet (see Jain, Durresi, and Paul (2011)) has stimulated research on pricing models (cf. Wolf et al. (2012)). In addition, there has been some progress made in the development of pricing models for various service-oriented Internets (cf. Laffont et al. (2003), Hermalin and Katz (2007), Zhang et al. (2010), Lv and Rouskas (2010), Musacchio et al. (2011), and Economides and Tag (2012)). A number of researchers focused on developing efficient dynamic pricing schemes to control congestion in the network. Singh et al. (2000) proposed a dynamic pricing framework that is implementable in a differentiated-service Internet architecture. Yaipairoj and Harmantzis (2004) presented a dynamic pricing model for mobile network services with QoS. Users can choose between offered alternatives based on their preferences when the system faces congestion. Moreover, it is imperative to study and analyze the underlying dynamics of the various economic decision-makers from content providers to transport network service providers in terms of both pricing as well as quality.

In this paper, we develop a projected dynamical systems (PDS) model of a service-oriented Internet. Such dynamical systems were introduced by Dupuis and Nagurney (1993) and have been used in a variety of applications from transportation, spatial economic and oligopolistic market problems (see Nagurney and Zhang (1996), Nagurney (1999), and the references therein) to supply chain network problems (cf. Nagurney (2006), Nagurney, Cruz, and Toyasaki (2008), and Cruz (2008), among others) and finance (see Nagurney (2008)). In addition, PDSs have been applied in population games by Sandholm (2010) and in neuroscience by Girad et al. (2008). More recently, PDSs have been utilized to capture the dynamics of oligopolistic competition with the inclusion of quality (see Nagurney and Li (2013)); to model the dynamics of a service-oriented Internet with only quality associated with content provision by Nagurney et al. (2013), and also to capture that associated with network provision by Nagurney and Wolf (2013). Here, for the first time, we model the dynamics of both price and quality competition of both content providers and of network providers. Our work is an attempt to complete both of the latter models in terms of price setting with the consideration of quality of service for both content and network provision.

The continuous-time dynamic model that we propose describes the evolution of the prices charged by the content providers and the network providers, as well as their quality levels of content and network transport provision, respectively. We provide qualitative results, including stability analysis, and also present a discrete-time algorithm for the iterative computation and tracking of the prices and quality levels until the stationary point, equivalently, the equilibrium state is achieved. This work extends and completes the static Internet network economic model of Saberi, Nagurney, and Wolf (2013) by describing the underlying dynamic behavior, accompanied by qualitative analysis, and with the provision of additional numerical examples.

This paper is organized as follows. In Section 2, we develop the model and describe the content providers' and the network providers' decision-making behaviors, and formulate the dynamics of the prices and the quality levels of the content and the network providers as a projected dynamical system (cf. Dupuis and Nagurney (1993), Zhang and Nagurney (1995), Nagurney and Zhang (1996), and Nagurney (2006)). We establish that the set of stationary points of the projected dynamical system coincides with the set of solutions to the derived variational inequality problem in Saberi, Nagurney, and Wolf (2013). The associated stability results are also provided. In Section 3, we present the algorithm to track the trajectories of the prices and quality levels over time until the equilibrium values are attained. We then apply the discrete-time algorithm to several numerical examples to further illustrate the model. We summarize our results and present our conclusions in Section 4, along with suggestions for future research.

2 The Dynamic Network Economic Model of a Service-Oriented Internet with Price and Quality Competition

In this Section, we develop the dynamic network economic model of a service-oriented Internet with price and quality competition. Unlike earlier models that focused on dynamics (cf. Nagurney et al. (2013) and Nagurney and Wolf (2013)), the new model allows for distinct quality levels associated with content provision and with transport network service provision. Moreover, we utilize direct demand functions, rather than inverse demand (price) functions, to capture the demand for content and network provision. Users (consumers) at the demand markets provide feedback to the content providers and the network providers in terms of the prices that they charge and their quality levels through the demands. Here, the demands are for the combination of content and network provision.

The network structure of the problem, which depicts the direction of the content flows, is given in Figure 1. Specifically, we assume m content providers, with a typical content provider denoted by CP_i; n network providers, which provide the transport of the content to the consumers at the demand markets, with a typical network provider denoted by NP_j, and o demand markets of users, with a typical demand market denoted by u_k.

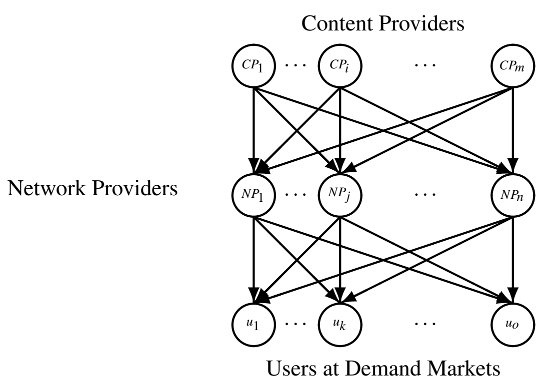

Fig. 1 The Network Structure of the Model's Content Flows

The notation for the model is given in Table 1. We first discuss what is meant by quality in the context of our model and describe specific functional forms, which are then utilized in the numerical examples. We then describe the behavior of the content providers and, subsequently, that of the network providers. We construct the projected dynamical system which formulates, in a unified manner, the dynamics of the content provider prices and quality levels and those of the network providers.

Table 1 Notation for the Dynamic Network Economic Model of a Service-Oriented Internet with Price and Quality Competition

Notation	Definition
p_{c_i}	the price CP_i; $i=1,\ldots,m$, charges the users for a unit of his content. The p_{c_i}; $i=1,\ldots,m$, are grouped into the vector $p_c \in R^m_+$.
p_{s_j}	the price NP_j; $j=1,\ldots,n$, charges the users for a unit of content transmitted by NP_j, with the p_{s_j}; $j=1,\ldots,n$, grouped into the vector $p_s \in R^n_+$.
p_{t_j}	the fixed transmission fee that NP_j; $j=1,\ldots,n$, charges the content providers for transmitting a unit of content.
q_{c_i}	the quality of CP_i's content. The q_{c_i}; $i=1,\ldots,m$, are grouped into the vector $q_c \in R^m_+$.
q_{s_j}	the quality of NP_j's transmission service. The q_{s_j}; $j=1,\ldots,n$, are grouped into the vector $q_s \in R^n_+$.
$d_{ijk}(p_c,q_c,p_s,q_s)$	the demand for CP_i's content; $i=1,\ldots,m$, transmitted by NP_j; $j=1,\ldots,n$, at demand market u_k; $k=1,\ldots,o$. The demand function d_{ijk} is monotonically decreasing (increasing) in p_{c_i} and p_{s_j} (q_{c_i} and q_{s_j}), and monotonically increasing (decreasing) in the other prices (quality levels).
SCP_i	the total supply of content of CP_i; $i=1,\ldots,m$.
TNP_j	the total amount of content transmitted by NP_j; $j=1,\ldots,n$.
$CC_i(SCP_i,q_{c_i})$	the total cost of CP_i; $i=1,\ldots,m$, to produce the content.
$CS_j(TNP_j,q_{s_j})$	the total cost of NP_j; $j=1,\ldots,n$, to maintain its network based on the total traffic passed through and its quality level.

2.1 Modeling of Quality in a Service-Oriented Internet

The quality of content provided can be specified for a specific domain of content, e.g., video streaming. In this case, quality is defined as the quality of videos produced by the content provider CP_i and the production cost CC_i is a convex and continuous function of quality of service as well as demand. Here we assume that the demand is equal to the supply, so that $CC_i = CC_i(SCP_i, q_{c_i})$. A possible functional form for CC_i is given by $K(SCP_i^2 + q_{c_i}^2)$. Of course, a special case of this functional form would be $Kq_{c_i}^2$, which would mean that the production cost of CP_i depends only on the quality of his product content.

The quality of the network transport service associated with NP_j, q_{s_j}, in turn, can be defined by various metrics such as the latency, jitter, or bandwidth. Latency is a measure of the delay that the traffic experiences as it traverses a network, and jitter is defined as the variation in that delay. Bandwidth is measured as the amount of data that can pass through a point in a network over time (see Smith and Garcia-Luna-Aceves (2008)). In our framework (see also Saberi, Nagurney, and

Wolf (2013)), we define the quality as the "expected delay," which is computed by the Kleinrock function (see Altman, Legout, and Xu (2011)) as the reciprocal of the square root of delay: $q_{s_j} = \frac{1}{\sqrt{Delay}} = \sqrt{b(d, q_{s_j}) - D}$, where $b(d, q_{s_j})$ is the total bandwidth of the network and is a function of demand d and quality, that is: $b(d, q_{s_j}) = d + q_{s_j}^2$. Hence, the greater the demand at higher quality, the larger the amount of bandwidth used. The network provider incurs a cost of transferring the demand while supporting q_{s_j} for data transport, denoted by CS_j. We assume a convex, continuous, and differentiable transfer function for NP_j of the following general form: $CS_j(TNP_j, q_{s_j}) = R\big(TNP_j + q_{s_j}^2\big)$, where R is the unit cost of bandwidth.

2.2 *The Behavior of the Content Providers and Their Price and Quality Dynamics*

Each CP_i produces distinct (but substitutable) content of specific quality q_{c_i}, and sells at a unit price p_{c_i}. The total supply of CP_i, SCP_i, is given by:

$$SCP_i = \sum_{j=1}^{n} \sum_{k=1}^{o} d_{ijk}, \quad i = 1, \ldots, m. \tag{1}$$

We assume that the content providers are profit-maximizers, where the profit or utility of CP_i, U_{CP_i}; $i = 1, \ldots, m$, which is the difference between his total revenue and his total cost, is given by the expression:

$$U_{CP_i}(p_c, q_c, p_s, q_s) = \sum_{j=1}^{n} (p_{c_i} - p_{t_j}) \sum_{k=1}^{o} d_{ijk} - CC_i(SCP_i, q_{c_i}). \tag{2}$$

Let $\mathscr{K}_i^1$ denote the feasible set corresponding to CP_i, where $\mathscr{K}_i^1 \equiv \{(p_{c_i}, q_{c_i}) \mid p_{c_i} \geq 0, \text{ and } q_{c_i} \geq 0\}$. Hence, the price charged by each CP_i and his quality level must be nonnegative. We assume that the utility functions in (2) for all i are continuous, continuously differentiable, and concave.

We now propose a dynamic adjustment process for the evolution of the content providers' prices and quality levels. In our framework, the rate of change of the price charged by CP_i; $i = 1, \ldots, m$, is in proportion to $\frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial p_{c_i}}$, as long as the price p_{c_i} is positive. Namely, when $p_{c_i} > 0$,

$$\dot{p}_{c_i} = \frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial p_{c_i}}, \tag{3}$$

where $\dot{p}_{c_i}$ denotes the rate of change of p_{c_i}. However, when $p_{c_i} = 0$, the nonnegativity condition on the price forces the price p_{c_i} to remain zero when $\frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial p_{c_i}} \leq 0$. Hence, in this case, we are only guaranteed of having possible increases in the price. Namely, when $p_{c_i} = 0$,

$$\dot{p}_{c_i} = \max\{0, \frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial p_{c_i}}\}. \quad (4)$$

Note that (4) is economically meaningful since when the marginal utility (profit) with respect to the price charged by CP_i is positive then we can expect the price that he charges for the content to increase; similarly, if the marginal utility (profit) with respect to the price that he charges is negative, then we can expect the price that he charges for the content to decrease. The max operator in (4) guarantees that the price will not take on a negative value, since it must satisfy the nonnegativity constraint.

We may write (3) and (4) concisely for each CP_i; $i = 1, \ldots, m$, as:

$$\dot{p}_{c_i} = \begin{cases} \frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial p_{c_i}}, & \text{if} \quad p_{c_i} > 0 \\ \max\{0, \frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial p_{c_i}}\}, & \text{if} \quad p_{c_i} = 0. \end{cases} \quad (5)$$

As for CP_i's quality level, when $q_{c_i} > 0$, then

$$\dot{q}_{c_i} = \frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial q_{c_i}}, \quad (6)$$

where $\dot{q}_{c_i}$ denotes the rate of change of q_{c_i}; otherwise:

$$\dot{q}_{c_i} = \max\{0, \frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial q_{c_i}}\}, \quad (7)$$

since q_{c_i} must be nonnegative.

Combining (6) and (7), we may write, for each CP_i; $i = 1, \ldots, m$:

$$\dot{q}_{c_i} = \begin{cases} \frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial q_{c_i}}, & \text{if} \quad q_{c_i} > 0 \\ \max\{0, \frac{\partial U_{CP_i}(p_c, q_c, p_s, q_s)}{\partial q_{c_i}}\}, & \text{if} \quad q_{c_i} = 0. \end{cases} \quad (8)$$

The system (8) is also economically meaningful, since we can expect the quality level associated with CP_i's content to increase (decrease) if the associated marginal utility (profit) is positive (negative). In addition, we are guaranteed that the quality of CP_i's content is never negative.

2.3 The Behavior of the Network Providers and Their Price and Quality Dynamics

Each NP_j; $j = 1, \ldots, n$, selects his quality q_{s_j} and the price p_{t_j} that he charges each content provider to transfer one unit of content to the users, and the price p_{s_j} that he charges users to transfer them one unit of content. Theoretically, every content provider is connected to every network provider and, subsequently, to all users, as depicted in Figure 1. However, solution of the model will determine which links have positive flows on them in terms of content. The total amount of content of services transported by NP_j, TNP_j, is given by:

$$TNP_j = \sum_{i=1}^{m} \sum_{k=1}^{o} d_{ijk}, \quad j = 1, \ldots, n. \tag{9}$$

The utility of NP_j; $j = 1, \ldots, n$, U_{NP_j}, corresponds to his profit and is the difference between his income and his cost, that is:

$$U_{NP_j}(p_c, q_c, p_s, q_s) = (p_{s_j} + p_{t_j})TNP_j - CS_j(TNP_j, q_{s_j}). \tag{10}$$

Let $\mathscr{K}_j^2$ denote the feasible set corresponding to network provider j, where $\mathscr{K}_j^2 \equiv \{(p_{s_j}, q_{s_j}) \mid p_{s_j} \geq 0, \text{and } q_{s_j} \geq 0\}$. Hence, NP_j's price and quality must both be nonnegative. The utility functions in (10) for all j are assumed to be continuous, continuously differentiable, and concave.

Although the network provider needs to determine the price to charge the content provider, p_{t_j}, he cannot maximize his utility with respect to p_{t_j} simultaneously with p_{s_j}. Note that the providers' utilities are linear functions of p_{t_j}, so that if p_{t_j} is under the control of one of the providers, it would simply be set at an extreme value and, subsequently, lead to zero demand and zero income. Therefore, p_{t_j} is assumed to be an exogenous parameter in this model.

We now describe the dynamics. Using similar arguments to those in Section 2.2, we have that the rate of change of the price for NP_j, $\dot{p}_{s_j}$; $j = 1, \ldots, n$, can be expressed as:

$$\dot{p}_{s_j} = \begin{cases} \frac{\partial U_{NP_j}(p_c, q_c, p_s, q_s)}{\partial p_{s_j}}, & \text{if} \quad p_{s_j} > 0 \\ \max\{0, \frac{\partial U_{NP_j}(p_c, q_c, p_s, q_s)}{\partial p_{s_j}}\}, & \text{if} \quad p_{s_j} = 0. \end{cases} \tag{11}$$

Analogously, for the quality level of NP_j; $j = 1, \ldots, n$, we may write:

$$\dot{q}_{s_j} = \begin{cases} \frac{\partial U_{NP_j}(p_c, q_c, p_s, q_s)}{\partial q_{s_j}}, & \text{if} \quad q_{s_j} > 0 \\ \max\{0, \frac{\partial U_{NP_j}(p_c, q_c, p_s, q_s)}{\partial q_{s_j}}\}, & \text{if} \quad q_{s_j} = 0. \end{cases} \tag{12}$$

Before proceeding to the construction of the projected dynamical systems model, we depict the financial payment flows associated with our dynamic network economic model in Figure 2. The directions of the arrows reflect the direction of the financial payments. The prices charged, in turn, would have the opposite direction to the associated financial payment.

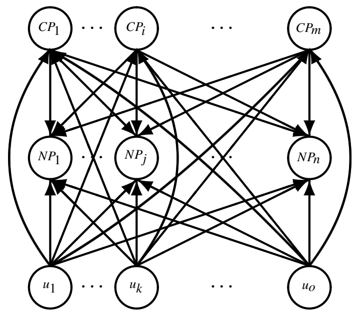

Fig. 2 The Network Structure of the Model's Financial Payment Flows

2.4 The Projected Dynamical System

Consider now the dynamic network economic model in which the content provider prices evolve according to (5) and their quality levels evolve according to (8). Similarly, the quality levels of the network providers evolve according to (12) and the prices that they charge according to (11). Let X denote the $(2m+2n)$-dimensional vector consisting of the vectors: (p_c, q_c, p_s, q_s). We also define the feasible set $\mathscr{K} \equiv \prod_{i=1}^{m} K_i^1 \times \prod_{j=1}^{n} K_j^2$. Finally, we define the $(2m+2n)$-dimensional vector $F(X)$ with components:

$$-\frac{\partial U_{CP_i}(p_c,q_c,p_s,q_s)}{\partial p_{c_i}}, -\frac{\partial U_{CP_i}(p_c,q_c,p_s,q_s)}{\partial q_{c_i}}; \quad i=1,\ldots,m;$$

$$-\frac{\partial U_{NP_j}(p_c,q_c,p_s,q_s)}{\partial p_{s_j}}, -\frac{\partial U_{NP_j}(p_c,q_c,p_s,q_s)}{\partial q_{s_j}}; \quad j=1,\ldots,n. \tag{13}$$

All vectors are assumed to be column vectors.

Then the dynamic model described above can be rewritten as the projected dynamical system (cf. Nagurney and Zhang (1996)) defined by the following initial value problem:

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \quad X(0) = X^0, \tag{14}$$

where $\Pi_{\mathscr{K}}$ is the projection operator of $-F(X)$ onto $\mathscr{K}$ and X^0 is the initial point $(p_c^0, q_c^0, p_s^0, q_s^0)$ corresponding to the initial price and quality levels of the content and the network providers. Specifically, according to Dupuis and Nagurney (1993), $\Pi_{\mathscr{K}}$ is the projection, with respect to $\mathscr{K}$, with $\mathscr{K}$ being a convex polyhedron, of the vector $-F(X)$ at X, defined as:

$$\Pi_{\mathscr{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathscr{K}}(X - \delta F(X)) - X}{\delta}, \tag{15}$$

with $P_{\mathscr{K}}$ being the projection map:

$$P_{\mathscr{K}}(X) = \text{argmin}_{z \in \mathscr{K}} \|X - z\|, \tag{16}$$

and where $\|\cdot\| = \langle x, x \rangle$. In our model, the projection operator takes on a nice explicit form because the feasible set $\mathscr{K}$ is the nonnegative orthant.

The trajectory associated with (14) provides the dynamic evolution of the prices charged and the quality levels of both the content providers and the network providers and the dynamic interactions among the content and the network providers and the users at the demand markets through the demand functions.

As emphasized in Nagurney and Zhang (1996), the dynamical system (14) is non-classical in that the right-hand side is discontinuous in order to guarantee that the constraints, that is, the nonnegativity assumption on all the prices and quality levels, are satisfied. Dupuis and Nagurney (1993) introduced such dynamical systems and they have been used, to-date, in numerous competitive applications, as noted in the Introduction. Here, for the first time, we model the dynamics of both price and quality competition of both content and network providers.

Stationary/Equilibrium Point

We now present the relationship between the stationary points of the projected dynamical system (14) and the solutions, commonly referred to as equilibria (cf. Nagurney (1999)), of the associated variational inequality problem: determine $X^* \in \mathscr{K}$ such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathscr{K}, \tag{17}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in n-dimensional Euclidean space, F is a continuous function from $\mathscr{K}$ to R^n, and $\mathscr{K}$ is closed and convex set.

Specifically, we have the following theorem, due to Dupuis and Nagurney (1993):

Theorem 1

The stationary points of the projected dynamical system (14), that is, those X^* that satisfy:

$$\dot{X} = 0 = \Pi_{\mathscr{K}}(X^*, -F(X^*)) \tag{18}$$

coincide with the solution of variational inequality (17).

Hence, we can immediately write down the variational inequality governing the equilibrium state (stationary point) associated with the above dynamic network economic model, in which no content provider nor any network provider has any incentive to alter his pricing and quality level strategies, as given below.

Corollary 1

$(p_c^*, q_c^*, p_c^*, q_c^*) \in \mathscr{K}$ is a stationary point of the projected dynamical system (14) if and only if it satisfies the variational inequality:

$$-\sum_{i=1}^{m} \frac{\partial U_{CP_i}(p_c^*, q_c^*, p_s^*, q_s^*)}{\partial p_{c_i}} \times (p_{c_i} - p_{c_i}^*) - \sum_{i=1}^{m} \frac{\partial U_{CP_i}(p_c^*, q_c^*, p_s^*, q_s^*)}{\partial q_{c_i}} \times (q_{c_i} - q_{c_i}^*)$$

$$-\sum_{j=1}^{n}\frac{\partial U_{NP_j}(p_c^*,q_c^*,p_s^*,q_s^*)}{\partial p_{s_j}}\times(p_{s_j}-p_{s_j}^*)$$

$$-\sum_{j=1}^{n}\frac{\partial U_{NP_j}(p_c^*,q_c^*,p_s^*,q_s^*)}{\partial q_{s_j}}\times(q_{s_j}-q_{s_j}^*)\geq 0,$$

$$\forall(p_c,q_c,p_s,q_s)\in\mathscr{K}, \tag{19}$$

or, equivalently,

$$\sum_{i=1}^{m}\Bigg[-\sum_{j=1}^{n}\sum_{k=1}^{o}d_{ijk}-\sum_{j=1}^{n}\sum_{k=1}^{o}\frac{\partial d_{ijk}}{\partial p_{c_i}}\times(p_{c_i}^*-p_{t_j})$$

$$+\frac{\partial CC_i(SCP_i,q_{c_i}^*)}{\partial SCP_i}\cdot\frac{\partial SCP_i}{\partial p_{c_i}}\Bigg]\times(p_{c_i}-p_{c_i}^*)$$

$$+\sum_{i=1}^{M}\Bigg[-\sum_{j=1}^{n}\sum_{k=1}^{o}\frac{\partial d_{ijk}}{\partial q_{c_i}}\times(p_{c_i}^*-p_{t_j})+\frac{\partial CC_i(SCP_i,q_{c_i}^*)}{\partial q_{c_i}}\Bigg]\times(q_{c_i}-q_{c_i}^*)$$

$$+\sum_{j=1}^{n}\Bigg[-\sum_{i=1}^{m}\sum_{k=1}^{o}d_{ijk}-\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial d_{ijk}}{\partial p_{s_j}}\times(p_{s_j}^*+p_{t_j})$$

$$+\frac{\partial CS_j(TNP_j,q_{s_j}^*)}{\partial TNP_j}\cdot\frac{\partial TNP_j}{\partial p_{s_j}}\Bigg]\times(p_{s_j}-p_{s_j}^*)$$

$$+\sum_{j=1}^{n}\Bigg[-\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial d_{ijk}}{\partial q_{s_j}}\times(p_{s_j}^*+p_{t_j})+\frac{\partial CS_j(TNP_j,q_{s_j}^*)}{\partial q_{s_j}}\Bigg]\times(q_{s_j}-q_{s_j}^*)\geq 0,$$

$$\forall(p_c,q_c,p_s,q_s)\in\mathscr{K}. \tag{20}$$

Variational inequalities (19) and (20) are precisely the ones obtained by Saberi, Nagurney, and Wolf (2013) for the static counterpart of our dynamic network economic model in which the content providers compete in price and quality until the Bertrand-Nash (cf. Bertrand (1883) and Nash (1950, 1951)) equilibrium is achieved whereby no content provider can improve upon his profits by altering his price and/or quality level. Similarly, the network providers also compete in price and quality until no network provider can improve upon his profits by altering his strategies and, hence, a Bertrand-Nash equilibrium is also achieved. Recall that a content price pattern and quality level pattern (p_c^*,q_c^*) is said to constitute a Bertrand-Nash equilibrium if for each content provider CP_i; $i=1,\ldots,m$:

$$U_{CP_i}(p_{c_i}^*,\hat{p_{c_i}^*},q_{c_i}^*,\hat{q_{c_i}^*},p_s^*,q_s^*,)\geq U_{CP_i}(p_{c_i},\hat{p_{c_i}^*},q_{c_i},\hat{q_{c_i}^*},p_s^*,q_s^*,),\quad\forall(p_{c_i},q_{c_i})\in\mathscr{K}_i^1,$$

where $\hat{p_{c_i}^*}\equiv(p_{c_1}^*,\ldots,p_{c_{i-1}}^*,p_{c_{i+1}}^*,\ldots,p_{c_m}^*)$ and $\hat{q_{c_i}^*}\equiv(q_{c_1}^*,\ldots,q_{c_{i-1}}^*,q_{c_{i+1}}^*,\ldots,q_{c_m}^*)$.

Similarly, a network price pattern and quality level pattern (p_s^*,q_s^*) is said to constitute a Bertrand-Nash equilibrium if for each network provider NP_j; $j=1,\ldots,n$:

$$U_{NP_j}(p_c^*, q_c^*, p_{s_j}^*, \hat{p_{s_j}^*}, q_{s_j}^*, \hat{q_{s_j}^*}) \geq U_{NP_j}(p_c^*, q_c^*, p_{s_j}, \hat{p_{s_j}^*}, q_{s_j}, \hat{q_{s_j}^*}), \quad \forall (p_{s_j}, q_{s_j}) \in \mathcal{K}_j^2,$$

where $\hat{p_{s_j}^*} \equiv (p_{s_1}^*, \ldots, p_{s_{j-1}}^*, p_{s_{j+1}}^*, \ldots, p_{s_n}^*)$ and $\hat{q_{s_j}^*} \equiv (q_{s_1}^*, \ldots, q_{s_{j-1}}^*, q_{s_{j+1}}^*, \ldots, q_{s_n}^*)$.

2.5 Stability Under Monotonicity

We now investigate whether, and under what conditions, the dynamic, continuous-time adjustment process defined by (14) approaches a stationary point/equilibrium. Recall that Lipschitz continuity of $F(X)$ (cf. Dupuis and Nagurney (1993) and Nagurney and Zhang (1996)) guarantees the existence of a unique solution to (14). In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X^0, \tag{21}$$

with $X^0(0) = X^0$. For convenience, we sometimes write $X^0 \cdot t$ for $X^0(t)$.

We propose the following definitions of stability for the adjustment process, which are adaptations of those introduced in Zhang and Nagurney (1995) (see also Nagurney and Zhang (1996)). We use $B(X, r)$ to denote the open ball with radius r and center X.

We now present some fundamental definitions, for completeness, and some basic qualitative results.

Definition 1
An equilibrium price and quality pattern X^* is stable, if for any $\varepsilon > 0$, there exists a $\delta > 0$, such that for all initial $X \in B(X^*, \delta)$ and all $t \geq 0$

$$X(t) \in B(X^*, \varepsilon). \tag{22}$$

The equilibrium point X^* is unstable, if it is not stable.

Definition 2
An equilibrium price and quality pattern X^* is asymptotically stable, if it is stable and there exists a $\delta > 0$ such that for all initial prices and qualities $X \in B(X^*, \delta)$

$$\lim_{t\to\infty} X(t) \longrightarrow X^*. \tag{23}$$

Definition 3
An equilibrium price and quality pattern X^* is globally exponentially stable, if there exist constants $b > 0$ and $\mu > 0$ such that

$$\|X^0(t) - X^*\| \leq b\|X^0 - X^*\| e^{-\mu t}, \quad \forall t \geq 0, \forall X^0 \in \mathcal{K}. \tag{24}$$

Definition 4
An equilibrium price and quality pattern X^* is a global monotone attractor, if the Euclidean distance $\|X(t)-X^*\|$ is nonincreasing in t for all $X \in \mathscr{K}$.

Definition 5
An equilibrium X^* is a strictly global monotone attractor, if $\|X(t)-X^*\|$ is monotonically decreasing to zero in t for all $X \in \mathscr{K}$.

We now investigate the stability of the dynamic adjustment process under various monotonicity conditions.

Recall (cf. Nagurney (1999)) that $F(X)$ is *monotone* if

$$\langle F(X)-F(X^*), X-X^*\rangle \geq 0, \quad \forall X, X^* \in \mathscr{K}. \tag{25}$$

$F(X)$ is *strictly monotone* if

$$\langle F(X)-F(X^*), X-X^*\rangle > 0, \quad \forall X, X^* \in \mathscr{K}, X \neq X^*. \tag{26}$$

$F(X)$ is *strongly monotone*, if there is an $\eta > 0$, such that

$$\langle F(X)-F(X^*), X-X^*\rangle \geq \eta\|X-X^*\|^2, \quad \forall X, X^* \in \mathscr{K}. \tag{27}$$

The monotonicity of a function F is closely related to the positive-definiteness of its Jacobian ∇F (cf. Nagurney (1999)). Specifically, if ∇F is positive-semidefinite, then F is monotone; if ∇F is positive-definite, then F is strictly monotone; and, if ∇F is strongly positive-definite, in the sense that the symmetric part of ∇F, $(\nabla F^T + \nabla F)/2$, has only positive eigenvalues, then F is strongly monotone.

In the context of our network economic model, where $F(X)$ is the vector of negative marginal utilities, we note that if the utility functions are twice differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions) for the model is positive-definite, then the corresponding $F(X)$ is strictly monotone.

We now present an existence and uniqueness result, the proof of which follows from the basic theory of variational inequalities (cf. Nagurney (1999)).

Theorem 2
Suppose that F is strongly monotone. Then there exists a unique solution to variational inequality (19); equivalently, to variational inequality (20).

We summarize in the following theorem the stability properties of the utility gradient process, under various monotonicity conditions on the marginal utilities.

Theorem 3
(i). *If $F(X)$ is monotone, then every stationary point of (14), provided its existence, is a global monotone attractor for the utility gradient process.*

(ii). *If $F(X)$ is strictly monotone, then there exists at most one stationary point / equilibrium of (14). Furthermore, given existence, the unique equilibrium is a strictly* *global monotone attractor for the utility gradient process.*

(iii). *If $F(X)$ is strongly monotone, then the stationary point / equilibrium of (14), which is guaranteed to exist, is also globally exponentially stable for the utility gradient process.*

Proof: The stability assertions follow from Theorems 3.5, 3.6, and 3.7 in Nagurney and Zhang (1996), respectively. The uniqueness in (ii) is a classical variational inequality result, whereas existence and uniqueness as in (iii) follows from Theorem 2. □

2.5.1 Example 1

We present Example 1 in order to illustrate some of the above concepts and results. The network consists of a single content provider, CP_1, a single network provider, NP_1, and users at a single demand market, u_1, as depicted in Figure 3.

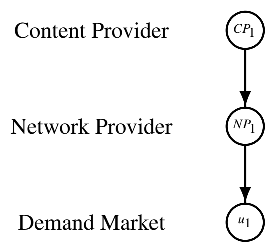

Fig. 3 Network Topology for Example 1

The data are as follows. The price p_{t_1} is 10. The demand function is:

$$d_{111} = 100 - .5p_{s_1} - .8p_{c_1} + .6q_{s_1} + .5q_{c_1}.$$

The cost functions of CP_1 and NP_1 are, respectively:

$$CC_1 = 2(d_{111}^2 + q_{c_1}^2), \qquad CS_1 = 2.2\big(d_{111} + q_{s_1}^2\big),$$

and their utility/profit functions are, respectively:

$$U_{CP_1} = (p_{c_1} - p_{t_1})d_{111} - 2(d_{111}^2 + q_{c_1}^2),$$

$$U_{NP_1} = (p_{s_1} + p_{t_1})d_{111} - 2.2(d_{111} + q_{s_1}^2).$$

Hence, we have that:

$$
\begin{aligned}
F_{p_{c_1}} &= -\frac{\partial U_{CP_1}}{\partial p_{c_1}} \\
&= -\frac{\partial\left[(p_{c_1}-10)d_{111}-2(d_{111}^2+q_{c_1}^2)\right]}{\partial p_{c_1}} \\
&= -[d_{111}+(p_{c_1}-10)\cdot\frac{\partial d_{111}}{\partial p_{c_1}}-4d_{111}\cdot\frac{\partial d_{111}}{\partial p_{c_1}}] \\
&= -[d_{111}+(p_{c_1}-10)\cdot(-.8)-4d_{111}\cdot(-.8)] \\
&= -4.2d_{111}+.8p_{c_1}-8 \\
&= -4.2\cdot(100-.5p_{s_1}-.8p_{c_1}+.6q_{s_1}+.5q_{c_1})+.8p_{c_1}-8 \\
&= -428+2.1p_{s_1}+4.16p_{c_1}-2.52q_{s_1}-2.1q_{c_1};
\end{aligned}
$$

$$
\begin{aligned}
F_{q_{c_1}} &= \frac{-\partial U_{CP_1}}{\partial q_{c_1}} \\
&= -\frac{\partial\left[(p_{c_1}-10)d_{111}-2(d_{111}^2+q_{c_1}^2)\right]}{\partial q_{c_1}} \\
&= -[(p_{c_1}-10)\cdot\frac{\partial d_{111}}{\partial q_{c_1}}-(4d_{111}\cdot\frac{\partial d_{111}}{\partial q_{c_1}}+4q_{c_1})] \\
&= -[(p_{c_1}-10).5-4d_{111}\cdot.5-4q_{c_1}] \\
&= 200-p_{s_1}-1.6p_{c_1}+1.2q_{s_1}+q_{c_1}+4q_{c_1}-.5p_{c_1}+5 \\
&= 205-p_{s_1}-2.1p_{c_1}+1.2q_{s_1}+5q_{c_1};
\end{aligned}
$$

$$
\begin{aligned}
F_{p_{s_1}} &= -\frac{\partial U_{NP_1}}{\partial p_{s_1}} \\
&= -\frac{\partial\left[(p_{s_1}+10)d_{111}-2.2(d_{111}+q_{s_1}^2)\right]}{\partial p_{s_1}} \\
&= -[d_{111}+(p_{s_1}+10)\cdot\frac{\partial d_{111}}{\partial p_{s_1}}-2.2\cdot\frac{\partial d_{111}}{\partial p_{s_1}}] \\
&= -[d_{111}+(p_{s_1}+10)\cdot(-.5)-2.2\cdot(-.5)] \\
&= -(100-.5p_{s_1}-0.8p_{c_1}+.6q_{s_1}+.5q_{c_1}-.5p_{s_1}-5+1.1) \\
&= -96.1+p_{s_1}+.8p_{c_1}-.6q_{s_1}-.5q_{c_1};
\end{aligned}
$$

$$
\begin{aligned}
F_{q_{s_1}} &= -\frac{\partial U_{NP_1}}{\partial q_{s_1}} \\
&= -\frac{\partial \left[(p_{s_1}+10)d_{111} - 2.2(d_{111}+q_{s_1}^2)\right]}{\partial q_{s_1}} \\
&= -\left[(p_{s_1}+10)\cdot \frac{\partial d_{111}}{\partial q_{s_1}} - (2.2\cdot \frac{\partial d_{111}}{\partial q_{s_1}} + 4.4 q_{s_1})\right] \\
&= -\left[(p_{s_1}+10)\cdot .6 - 2.2\cdot .6 - 4.4 q_{s_1}\right] \\
&= -4.68 - .6 p_{s_1} + 4.4 q_{s_1}.
\end{aligned}
$$

The Jacobian matrix of -$\nabla U(p_{c_1}, q_{c_1}, p_{s_1}, q_{s_1})$, denoted by $J(p_{c_1}, q_{c_1}, p_{s_1}, q_{s_1})$, is

$$
J(p_{c_1}, q_{c_1}, p_{s_1}, q_{s_1}) = \begin{pmatrix} 4.16 & -2.1 & 2.1 & -2.52 \\ -2.1 & 5 & -1 & 1.2 \\ .8 & -.5 & 1 & -.6 \\ 0 & -.6 & 0 & 4.4 \end{pmatrix}.
$$

Since the symmetric part of $J(p_{c_1}, q_{c_1}, p_{s_1}, q_{s_1})$, $(J^T+J)/2$, has only positive eigenvalues, which are: .43, 2.40, 4.03, and 7.70, the $F(X)$ in Example 1 (cf. (13)) is strongly monotone. Thus, according to Theorem 3, there exists a unique equilibrium, which is also globally exponentially stable for the utility gradient process. In the next Section, we compute the equilibrium solution to this and other numerical examples.

3 The Algorithm and Numerical Examples

Note that, for computation purposes, we need to identify a discrete-time adjustment process or algorithm which will track the continuous-time process (14) until a stationary point is achieved (equivalently, an equilibrium point). In this Section, we recall the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993) and provides us with such a computational procedure. Specifically, iteration τ of the Euler method is given by:

$$
X^{\tau+1} = P_{\mathscr{K}}(X^{\tau} - a_{\tau} F(X^{\tau})). \tag{28}
$$

As shown in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996), for convergence of the general iterative scheme, which induces the Euler method, among other methods, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$. Specific conditions for convergence of this scheme can be found for a variety of network based problems, similar to those constructed in Nagurney and Zhang (1996) and the references therein.

Explicit Formulae for the Euler Method Applied to the Service-oriented Internet with Price and Quality Competition

The elegance of this procedure for the computation of solutions to our network economic model of a service-oriented Internet can be seen in the following explicit formulae. Indeed, (28) yields the following closed form expressions for the price and the quality of each content and network provider $i = 1,\dots,m$; $j = 1,\dots,n$:

$$p_{c_i}^{\tau+1} = \max\Big\{0, p_{c_i}^{\tau} + a_\tau\big(\sum_{j=1}^{n}\sum_{k=1}^{o} d_{ijk} + \sum_{j=1}^{n}\sum_{k=1}^{o} \frac{\partial d_{ijk}}{\partial p_{c_i}} \times (p_{c_i}^{\tau} - p_{t_j}) - \frac{\partial CC_i(SCP_i, q_{c_i}^{\tau})}{\partial SCP_i} \cdot \frac{\partial SCP_i}{\partial p_{c_i}}\big)\Big\}, \quad (29)$$

$$q_{c_i}^{\tau+1} = \max\Big\{0, q_{c_i}^{\tau} + a_\tau\big(\sum_{j=1}^{n}\sum_{k=1}^{o} \frac{\partial d_{ijk}}{\partial q_{c_i}} \times (p_{c_i}^{\tau} - p_{t_j}) - \frac{\partial CC_i(SCP_i, q_{c_i}^{\tau})}{\partial q_{c_i}}\big)\Big\}, \quad (30)$$

$$p_{s_j}^{\tau+1} = \max\Big\{0, p_{s_j}^{\tau} + a_\tau\big(\sum_{i=1}^{m}\sum_{k=1}^{o} d_{ijk} + \sum_{i=1}^{m}\sum_{k=1}^{o} \frac{\partial d_{ijk}}{\partial p_{s_j}} \times (p_{s_j}^{\tau} + p_{t_j}) - \frac{\partial CS_j(TNP_j, q_{s_j}^{\tau})}{\partial TNP_j} \cdot \frac{\partial TNP_j}{\partial p_{s_j}}\big)\Big\}, \quad (31)$$

$$q_{s_j}^{\tau+1} = \max\Big\{0, q_{s_j}^{\tau} + a_\tau\big(\sum_{i=1}^{m}\sum_{k=1}^{o} \frac{\partial d_{ijk}}{\partial q_{s_j}} \times (p_{s_j}^{\tau} + p_{t_j}) - \frac{\partial CS_j(TNP_j, q_{s_j}^{\tau})}{\partial q_{s_j}}\big)\Big\}. \quad (32)$$

Note that all the functions to the right of the equal signs in (29) – (32) are evaluated at their respective variables computed at the τ-th iteration.

We now provide the convergence result. The proof is direct from Theorem 5.8 in Nagurney and Zhang (1996).

Theorem 4: Convergence
In the service-oriented Internet network economic problem, assume that $F(X) = -\nabla U(p_c, q_c, p_s, q_s)$ is strongly monotone. Also, assume that F is uniformly Lipschitz continuous. Then there exists a unique equilibrium price and quality pattern $(p_c^*, q_c^*, p_s^*, q_s^*) \in \mathcal{K}$ and any sequence generated by the Euler method as given by (29) - (32), where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$ converges to $(p_c^*, q_c^*, p_s^*, q_s^*)$ satisfying (20); equivalently, (18).

We implemented the Euler method to compute solutions to service-oriented Internet network economic problems in Matlab. The Euler method was deemed to have converged if, at a given iteration, the absolute value of the difference of each price and each quality level differed from its respective value at the preceding iteration by no more than $\varepsilon = 10^{-6}$. The sequence $\{a_\tau\}$ used was: $.1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \dots)$. We initialized the algorithm by setting $p_{c_i}^0 = q_{c_i}^0 = p_{s_j}^0 = q_{s_j}^0 = 0, \quad \forall i, j$.

Example 1 Revisited
We first applied the Euler method to compute the equilibrium prices and quality levels for Example 1.

The Euler method required 136 iterations for convergence to the computed equilibrium:

$$p^*_{c_1} = 94.50, \quad q^*_{c_1} = 2.51 \quad p^*_{s_1} = 24.40, \quad q^*_{s_1} = 4.38,$$

with an incurred demand of $d_{111} = 16.10$.

The utility/profit of CP_1 is 829.32 and that of NP_1: 475.70.

If we change p_{t_1} to 0, then the new equilibrium is:

$$p^*_{c_1} = 35.39, \quad q^*_{c_1} = 2.59, \quad p^*_{s_1} = 87.14, \quad q^*_{s_1} = 4.52,$$

with an incurred demand of $d_{111} = 16.08$.

The utility/profit of CP_1 is now 882.01 and that of NP_1 is 505.92.

Hence, in this example, NP_1 would be better off in terms of his profit, if he does not charge CP_1, that is, $p_{t_1} = 0$ since the users are more sensitive to the content provider's price.

Example 2
In Example 2, there are 2 content providers, CP_1 and CP_2, a single network provider, NP_1, and users at a single demand market, u_1, as depicted in Figure 4.

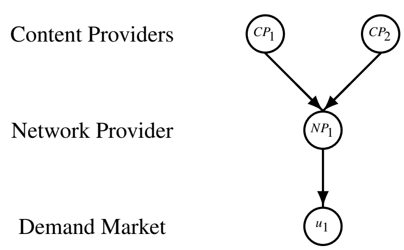

Fig. 4 Network Topology for Example 2

The data are as follows. The demand functions are:

$$d_{111} = 100 - 1.6p_{c_1} + .65p_{c_2} - 1.35p_{s_1} + 1.2q_{c_1} - .42q_{c_2} + 1.54q_{s_1},$$

$$d_{211} = 112 + .65p_{c_1} - 1.5p_{c_2} - 1.35p_{s_1} - .42q_{c_1} + 1.3q_{c_2} + 1.54q_{s_1}.$$

The cost functions of the content providers are:

$$CC_1 = 1.7q^2_{c_1}, \quad CC_2 = 2.4q^2_{c_2}$$

and their utilities/profit functions are:

$$U_{CP_1} = (p_{c_1} - p_{t_1})d_{111} - CC_1, \qquad U_{CP_2} = (p_{c_2} - p_{t_1})d_{211} - CC_2.$$

The cost function of the network provider is:

$$CS_1 = 2.1(d_{111} + d_{211} + q_{s_1}^2)$$

and its utility/profit function is:

$$U_{NP_1} = (p_{s_1} + p_{t_1})(d_{111} + d_{211}) - CS_1.$$

p_{t_1} is assumed to be 10.

The Jacobian matrix of -$\nabla U(p_{c_1}, q_{c_1}, p_{c_2}, q_{c_2}, p_{s_1}, q_{s_1})$, denoted by $J(p_{c_1}, q_{c_1}, p_{c_2}, q_{c_2}, p_{s_1}, q_{s_1})$, is

$$J(p_{c_1}, q_{c_1}, p_{c_2}, q_{c_2}, p_{s_1}, q_{s_1}) = \begin{pmatrix} 3.2 & -1.2 & -.65 & .42 & 1.35 & -1.54 \\ -1.2 & 3.4 & 0 & 0 & 0 & 0 \\ -.65 & .42 & 3 & -1.3 & 1.35 & -1.54 \\ 0 & 0 & -1.3 & 4.8 & 0 & 0 \\ .95 & -.78 & .85 & -.88 & 5.4 & -3.08 \\ 0 & 0 & 0 & 0 & -3.08 & 4.2 \end{pmatrix}.$$

Since the symmetric part of $J(p_{c_1}, q_{c_1}, p_{c_2}, q_{c_2}, p_{s_1}, q_{s_1})$, $(J^T + J)/2$, has only positive eigenvalues, which are 1.52, 1.61, 2.37, 4.22, 5.61, and 8.67, the $F(X)$ in Example 2 is strongly monotone. Thus, according to Theorem 3, there exists a unique equilibrium, which is also globally exponentially stable for the utility gradient process.

The Euler method converged in 2341 iterations to the following solution:

$$p_{c_1}^* = 51.45, \quad p_{c_2}^* = 56.75, \quad p_{s_1}^* = 42.64,$$

$$q_{c_1}^* = 14.63, \quad q_{c_2}^* = 12.66, \quad q_{s_1}^* = 37.06,$$

with incurred demands of:

$$d_{111} = 66.32, \quad d_{211} = 70.13.$$

The utility/profit of CP_1 is 2385.21 and of CP_2: 2894.58. The utility/profit of NP_1 is 4011.92.

Example 3

In Example 3, there is a single content provider, CP_1, two network providers, NP_1 and NP_2, and a single demand market, u_1, as depicted in Figure 5.

The demand functions are:

$$d_{111} = 100 - 1.7p_{c_1} - 1.5p_{s_1} + .8p_{s_2} + 1.76q_{c_1} + 1.84q_{s_1} - .6q_{s_2},$$

$$d_{121} = 100 - 1.7p_{c_1} + .8p_{s_1} - 1.8p_{s_2} + 1.76q_{c_1} - .6q_{s_1} + 1.59q_{s_2}.$$

The cost function of CP_1 is:

$$CC_1 = 1.5(d_{111} + d_{121} + q_{c_1}^2)$$

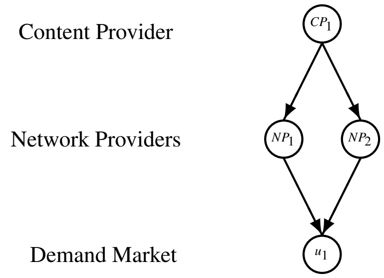

Fig. 5 Network Topology for Example 3

and its utility/profit function is:

$$U_{CP_1} = (p_{c_1} - p_{t_1})d_{111} + (p_{c_1} - p_{t_2})d_{121} - CC_1.$$

The network providers' cost functions are:

$$CS_1 = 1.8(d_{111} + q_{s_1}^2), \quad CS_2 = 1.7(d_{121} + q_{s_2}^2),$$

with their utility/profit functions given by:

$$U_{NP_1} = (p_{s_1} + p_{t_1})d_{111} - CS_1, \qquad U_{NP_2} = (p_{s_2} + p_{t_2})d_{121} - CS_2.$$

We set $p_{t_1} = 10$ and $p_{t_2} = 7$.

The Jacobian matrix of -$\nabla U(p_{c_1}, q_{c_1}, p_{s_1}, q_{s_1}, p_{s_2}, q_{s_2})$, denoted by $J(p_{c_1}, q_{c_1}, p_{s_1}, q_{s_1}, p_{s_2}, q_{s_2})$, is

$$J(p_{c_1}, q_{c_1}, p_{s_1}, q_{s_1}, p_{s_2}, q_{s_2}) = \begin{pmatrix} 6.8 & -3.52 & .7 & -1.24 & 1 & -.99 \\ -3.52 & 3 & 0 & 0 & 0 & 0 \\ 1.7 & -1.76 & 3 & -1.84 & -0.8 & .6 \\ 0 & 0 & -1.84 & 3.6 & 0 & 0 \\ 1.7 & -1.76 & -.8 & .6 & 3.6 & -1.59 \\ 0 & 0 & 0 & 0 & -1.59 & 3.4 \end{pmatrix}.$$

The symmetric part of $J(p_{c_1}, q_{c_1}, p_{s_1}, q_{s_1}, p_{s_2}, q_{s_2})$, $(J^T + J)/2$, has only positive eigenvalues, which are .66, 1.32, 1.84, 3.96, 5.85, and 9.77. Hence, the $F(X)$ in Example 3 is also strongly monotone and we know from Theorem 3, that there exists a unique equilibrium, which is also globally exponentially stable for the utility gradient process.

The Euler method required 120 iterations for convergence. The computed equilibrium solution is:

$$p_{c_1}^* = 64.90, \quad p_{s_1}^* = 57.98, \quad p_{s_2}^* = 43.24,$$

$$q_{c_1}^* = 64.41, \quad q_{s_1}^* = 33.82, \quad q_{s_2}^* = 22.70,$$

with incurred demands of:

$$d_{111} = 99.28, \quad d_{121} = 87.38.$$

The utility/profit of CP_1 is 4006.15. The utilities/profits of NP_1 and NP_2 are 4511.38, and 3366.23, respectively.

Example 4
In Example 4, there are two content providers, CP_1 and CP_2, two network providers, NP_1 and NP_2, and two markets of users, u_1 and u_2, as depicted in Figure 6.

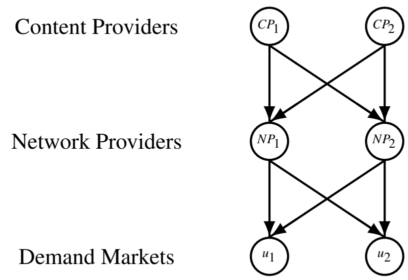

Fig. 6 Network Topology for Example 4

The demand functions are:

$$d_{111} = 100 - 2.1p_{c_1} + .5p_{c_2} - 2.3p_{s_1} + .6p_{s_2} + .63q_{c_1} - .4q_{c_2} + .62q_{s_1} - .4q_{s_2},$$

$$d_{112} = 112 - 2.2p_{c_1} + .5p_{c_2} - 2.4p_{s_1} + .6p_{s_2} + .75q_{c_1} - .4q_{c_2} + .56q_{s_1} - .4q_{s_2},$$

$$d_{121} = 100 - 2.1p_{c_1} + .5p_{c_2} + .6p_{s_1} - 2.2p_{s_2} + .63q_{c_1} - .4q_{c_2} - .4q_{s_1} + .59q_{s_2},$$

$$d_{122} = 112 - 2.2p_{c_1} + .5p_{c_2} + .6p_{s_1} - 2.1p_{s_2} + .75q_{c_1} - .4q_{c_2} - .4q_{s_1} + .68q_{s_2},$$

$$d_{211} = 110 + .5p_{c_1} - 2.3p_{c_2} - 2.3p_{s_1} + .6p_{s_2} - .4q_{c_1} + .76q_{c_2} + .62q_{s_1} - .4q_{s_2},$$

$$d_{212} = 104 + .5p_{c_1} - 2.05p_{c_2} - 2.4p_{s_1} + .6p_{s_2} - .4q_{c_1} + .61q_{c_2} + .56q_{s_1} - .4q_{s_2},$$

$$d_{221} = 110 + .5p_{c_1} - 2.3p_{c_2} + .6p_{s_1} - 2.2p_{s_2} - .4q_{c_1} + .76q_{c_2} - .4q_{s_1} + .59q_{s_2},$$

$$d_{222} = 104 + .5p_{c_1} - 2.05p_{c_2} + .6p_{s_1} - 2.1p_{s_2} - .4q_{c_1} + .61q_{c_2} - .4q_{s_1} + .68q_{s_2}.$$

The cost functions of the content providers are:

$$CC_1 = 3.7(q_{c_1}^2), \quad CC_2 = 5.1(q_{c_2}^2),$$

and their profit functions are, respectively:

$$U_{CP_1} = (p_{c_1} - p_{t_1})(d_{111} + d_{112}) + (p_{c_1} - p_{t_2})(d_{121} + d_{122}) - CC_1,$$

$$U_{CP_2} = (p_{c_2} - p_{t_1})(d_{211} + d_{212}) + (p_{c_2} - p_{t_2})(d_{221} + d_{222}) - CC_2.$$

The network providers' cost functions are:

$$CS_1 = 4.1(d_{111} + d_{112} + d_{211} + d_{212} + q_{s_1}^2), \quad CS_2 = 3.9(d_{121} + d_{122} + d_{221} + d_{222} + q_{s_2}^2),$$

and their profit functions are:

$$U_{NP_1} = (p_{s_1} + p_{t_1})(d_{111} + d_{112} + d_{211} + d_{212}) - CS_1,$$

$$U_{NP_2} = (p_{s_2} + p_{t_2})(d_{121} + d_{122} + d_{221} + d_{222}) - CS_2.$$

We set $p_{t_1} = 23$, and $p_{t_2} = 22$.

The Jacobian matrix of -$\nabla U(p_{c_1}, q_{c_1}, p_{c_2}, q_{c_2}, p_{s_1}, q_{s_1}, p_{s_2}, q_{s_2})$, denoted by $J(p_{c_1}, q_{c_1}, p_{c_2}, q_{c_2}, p_{s_1}, q_{s_1}, p_{s_2}, q_{s_2})$, is

$$J = \begin{pmatrix} 17.2 & -2.76 & -2 & 1.6 & 3.5 & -.38 & 3.1 & -.47 \\ -2.76 & 7.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1.6 & 17.4 & -2.74 & 3.5 & -.38 & 3.1 & -.47 \\ 0 & 0 & -2.74 & 10.2 & 0 & 0 & 0 & 0 \\ 3.3 & -.58 & 3.35 & -.57 & 18.8 & -2.36 & -2.4 & 1.6 \\ 0 & 0 & 0 & 0 & -2.36 & 8.2 & 0 & 0 \\ 3.3 & -.58 & 3.35 & -.57 & -2.4 & 1.6 & 17.2 & -2.54 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.54 & 7.8 \end{pmatrix}.$$

The symmetric part of $J(p_{c_1}, q_{c_1}, p_{c_2}, q_{c_2}, p_{s_1}, q_{s_1}, p_{s_2}, q_{s_2})$, $(J^T + J)/2$, has only positive eigenvalues, which are 6.54, 7.01, 7.57, 8.76, 10.24, 20.39, 20.94, and 22.75. Hence, the $F(X)$ in Example 4 is also strongly monotone and we know that the equilibrium solution is unique. The Euler method required 189 iterations for convergence, yielding:

$$p_{c_1}^* = 41.52, \quad p_{c_2}^* = 40.93, \quad p_{s_1}^* = 0.0, \quad p_{s_2}^* = 0.58,$$

$$q_{c_1}^* = 7.09, \quad q_{c_2}^* = 4.95, \quad q_{s_1}^* = 5.44, \quad q_{s_2}^* = 6.08,$$

with incurred demands of:

$$d_{111} = 37.04, \quad d_{112} = 45.42, \quad d_{121} = 35.91, \quad d_{122} = 45.21,$$

$$d_{211} = 38.83, \quad d_{212} = 42.00, \quad d_{221} = 37.70, \quad d_{222} = 41.79.$$

The profits of the content providers are, respectively, 2924.52 and 2828.79, and that of the network providers: 2964.97 and 2855.11.

Please refer to Figures 7, 8, and 9 to view the trajectories of the prices and the quality levels generated by the Euler method at iterations 0, 10, 20,..., 180, 189.

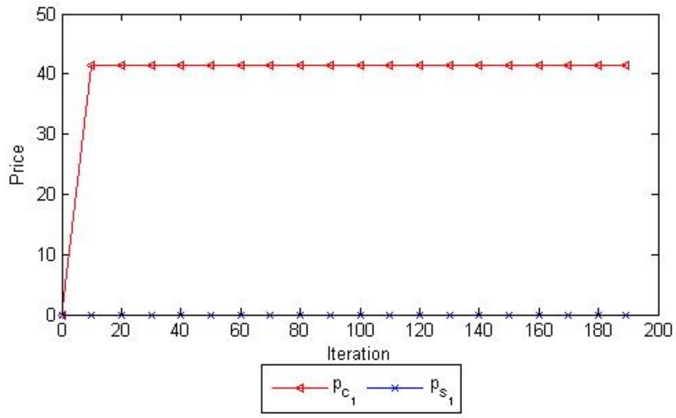

Fig. 7 Prices of Content Provider 1 and Network Provider 1 for Example 4

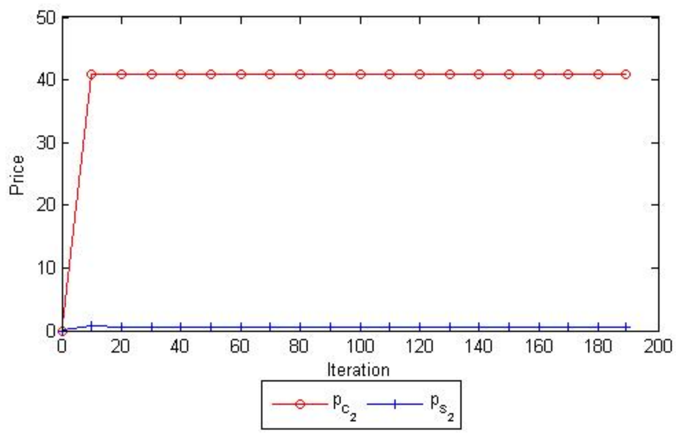

Fig. 8 Prices of Content Provider 2 and Network Provider 2 for Example 4

4 Summary and Conclusions

In this paper, we developed a new dynamic network economic model of a service-oriented FGI. The model handles price and quality competition among the content providers, who provide Internet services, and among the network providers, who transport the Internet services. Consumer direct demand functions that depend on the prices and the quality levels of both content and network providers are utilized, rather than their inverses, which allows for prices as strategic variables. The framework yields insights into the evolutionary processes of quality selection and the pricing of Internet services.

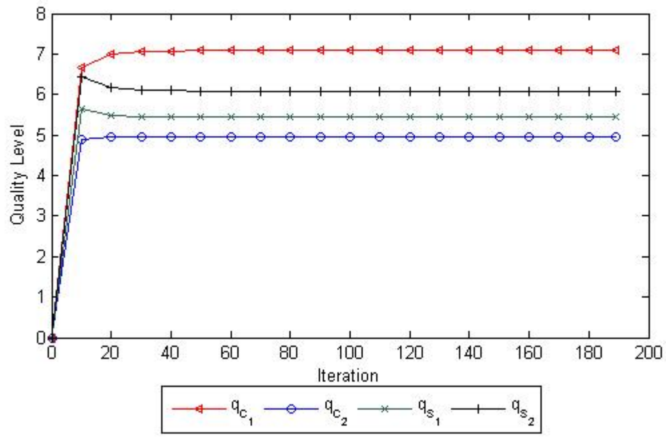

Fig. 9 Quality Levels of Content Providers and Network Providers for Example 4

Specifically, the projected dynamical systems model that we constructed provides a continuous-time adjustment process of the content providers' and the network providers' prices and quality levels, and guarantees that prices and quality levels remain nonnegative, as required by the constraints. The set of equilibrium/stationary points coincides with the set of solutions to the associated variational inequality problem. Qualitative properties, including stability analysis results, are also provided.

We proposed the Euler method, which provides a discretization of the continuous-time adjustment process and yields closed form expressions for the prices and the quality levels at each iteration step. This algorithm also tracks the values of the prices and quality levels over time until the equilibrium point is achieved. Convergence results were also given. The generality and practicality of our model and the computational procedure are illustrated through several numerical examples.

The FGI, as an exciting new area of research, is full of additional questions for investigation, some of which are identified below.

- The price mechanisms used in our model are usage-based with bandwidth-based pricing for the content and network providers. What would be the equilibrium outcomes if a flat-rate or a two-part tariff pricing mechanism would be applied instead? Would such pricing mechanisms increase the users' demand?
- Since long-term contracts lock in consumers, and have low flexibility, it would be interesting to consider short-term contracts, which might enable users to select among the service offerings from different providers, in a more dynamic manner. How would the pricing dynamics change in an FGI with short-term contracts?
- In our model, content providers and network providers have no restrictions on their services, with the exception that the prices that they charge and their service quality levels must be nonnegative. However, providers in an FGI might

be faced with some additional restrictions, that is, constraints. For example, what would be the dynamics and the equilibrium prices and quality levels for a content provider with a production capacity limitation? To what extent would the equilibrium price and quality level of a network provider with capacity restrictions for data transmission change in comparison with the case with no such limitations? Presently, we handled capacity limitations through the nonlinearity of the underlying cost functions, which can capture "congestion." In addition, we might wish to consider an upper bound or a non-zero lower bound for the quality level of a content or network provider's services. A non-zero, but positive, lower bound on the quality level, for example, might occur due to an imposed governmental regulation.

- Empirical studies could be used to validate our model and to yield a parameterization of our model that matches a practical FGI scenario..

We believe that the framework constructed in this paper can serve as the foundation to address the above issues in future research.

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