Defense Critical Supply Chain Networks and Risk Management with the Inclusion of Labor: Dynamics and Quantification of Performance and the Ranking of Nodes and Links

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Abstract: The efficient and effective performance of defense critical supply chain networks is essential to both national and global security. Disruptions to supply chains, heightened in the COVID-10 pandemic, and now further exacerbated because of growing geopolitical and other risks, as well as Russia's war against Ukraine, have garnered the attention of decision-makers and policy-makers, including those in the defense sector. In the paper, a rigorous methodological framework is presented for defense critical supply chain networks in the form of a defense supply chain network economy that captures the behavior of defense firms, which care about revenues as well as risk, and which includes the important labor resources and associated constraints. Variational inequality theory is used to provide alternative formulations of the governing Nash Equilibrium conditions, with a dynamic model counterpart used for the construction of an easy to implement algorithm that yields closed form expressions at each iteration of the defense product path flows and the Lagrange multipliers associated with the bounds on labor hours available on supply chain links. A defense supply chain network efficiency /performance measure is proposed and an associated importance indicator for supply chain network components. A resilience measure is also given that quantifies the resilience of the defense supply chain network economy to disruptions in labor. The modeling and algorithmic framework, as well as the measures proposed, are then illustrated via numerical examples.

Keywords: Defense; Supply Chains; Networks; Resilience; Labor; Game Theory; Variational Inequalities

1. Introduction

In February 2022, the U.S Department of Defense (DoD) issued a long-awaited report, "Securing Defense-Critical Supply Chains" (U.S. Department of Defense (2022)). The report was in response to Executive Order (E.O.) 14017, "America's Supply Chains," signed by President Joseph R. Biden Jr., to identify how to improve supply chain resilience and how to protect against material shortages, which had clearly become exacerbated in the COVID-19 pandemic (see Biden Jr. (2021), United States White House (2021)). The DoD's report provided an assessment of defense critical supply chains in order to improve the department's capacity to defend the United States. With the geopolitical risk rising globally and, with the war of Russia against Ukraine raging (cf. Bilefsky, Perez-Pena, and Nagurney and Ermagun (2022)), following the major invasion, beginning February 24, 2022, having a framework for the modeling, analysis, and solution of defense critical supply chains is of major importance. Of additional relevance is having a framework to identify which of the nodes and links, corresponding, for example, to manufacturing sites and processes, storage facilities, transportation and distribution, are important, since focusing on those can help to preserve the performance of the supply chain networks for critical defense products in the case of disruptions.

Parallel to the COVID-19 pandemic, which is a global healthcare disaster, not limited in location or to a time window, the number of disasters, including "natural" disasters, has been growing as well as the people affected by them (see Nagurney and Qiang (2009) and Kotsireas et al. (2021)). Hence, research on supply chain network performance and resilience has been garnering increasing attention (see Sheffi (2015), Ivanov and Dolgui (2020), Nagurney and Ermagun (2022), Novoszel and Wakolbinger (2022), Ramakrishnan (2022)) with supply chains networks for defense products being essential to national and, even global, security. In the DoD report, manufacturing, as well as the workforce, are considered to be strategic enablers and critical to building overall supply chain resilience.

In this paper, a defense critical supply chain network game theory model is constructed in which the defense firms compete noncooperatively in producing, transporting, storing, and distributing their substitutable defense products, which are distinguished by firm or "brand." Defense products could include weaponry, radars, tanks, or even life-saving vests and medical kits. The objective function faced by a defense firm that it wishes to maximize consists of the profit and the weighted total risk associated with its supply chain network. A crucial element of the model is the availability of labor associated with each supply chain network link and a bound on the labor hours available. The governing equilibrium concept is that of a Nash Equilibrium (1950, 1951). Under appropriate delineated assumptions, the governing equilibrium conditions are shown to satisfy a variational inequality problem for which existence of a solution is guaranteed. An alternative variational inequality is then constructed with both defense product path flows and Lagrange multipliers associated with the link labor bounds as variables, and with the underlying feasible set being the nonnegative orthant. A dynamic adjustment process is then proposed utilizing the theory of projected dynamical systems (see Nagurney and Zhang (1996)) and a discrete-time algorithm outlined for computational purposes. Here, we consider a defense supply chain network economy in that the defense demand markets, which can be associated with different governments, can procure the defense products from the defense firms, which can be in different countries. The supply chain network economy can correspond, for example, to defense firms associated with NATO, or the European Union, or other such organizational bodies.

We, subsequently, turn to the construction of a defense supply chain network efficiency / performance measure, which is then applied to define the importance of a network component, whether a node, a link, or a composition of nodes and links thereof. Note is then made of how the measure can be applied to measure resilience of the supply chain network to disruptions in labor. The inclusion of labor into general supply chain networks is a recent contribution, and was motivated by the impacts of the COVID-19 pandemic on workers, their health, loss of productivity, etc., as well as the negative effects of shortages of labor on profits as well as consumers. Towards that end, Nagurney (2021a,b) introduced labor into supply chain networks, beginning with optimization models, and then evolving to game theory models, with the model by Nagurney (2021c) being the most relevant to the one constructed in this paper. Here, however, we introduce risk, since risk is a characteristic of many supply chains these days, due to a challenging geopolitic landscape (see also Nagurney et al. (2005), Tang (2006), Tang and Tomlin (2009), Qiang, Nagurney, and Dong (2009), Wu and Blackhurst (2009), Kotsireas et al. (2021)). Furthermore, defense critical products can include high tech elements such as computer chips, which have been in short supply, as well as other raw materials that may be located in places under governance by antagonistic regimes.

The paper is organized as follows. In Section 2, the defense critical supply chain network game theory model with labor is presented and alternative variational inequality formulations given. In addition, a dynamic version of the model is constructed, whose set of stationary points coincides with the set of solutions to the variational inequality with defense product path flows and link Lagrange multipliers associated with the labor bounds as variables. A time-discretization of the continuous time adjustment processes, in the form of a discrete-algorithm, is provided. The algorithm is applied in Section 4 to illustrate the framework presented here in a series of defense critical supply chain network examples. In Section 3, the defense supply chain network efficiency /performance measure is proposed, along with the definition of the importance of a supply chain network component. In addition, the quantification of resilience of the supply chain network to labor disruptions is highlighted. A summary of results, along with the conclusions, and suggestions for future research, are provided in the Section 5.

2. Defense Critical Supply Chain Network Game Theory Modeling

The supply chain network model with labor constructed here focuses on defense critical products. We consider I firms involved in the production, transportation, storage, and ultimate distribution of the defense products, which are substitutable. The products could, for example, be related weaponry, such as missiles, or tanks, or even protective equipment such as helmets, life-saving vests for the military and/or citizens, or medical kits. The demand markets, here, represent the governmental defense demand markets. Note that we do not limit the model to a specific country. The demand markets can correspond to demand markets of different countries but they are assumed to be partners and not antagonists. Hence, this model could be useful, for example, for NATO countries, for countries in the European Union, or other such coalitions.

The topology of the supply chain networks of the firms in the "defense supply chain network economy" is depicted in Figure 1. All vectors are column vectors. The model builds upon the model in Nagurney (2021c) but with the addition of the crucial feature of risk management. Here we also focus on labor bounds on links and we provide, for the first time, a network performance/efficiency measure for a supply chain network game theory model with labor and a formalism for the identification of the importance of nodes and links and their ranking.

A typical defense firm is denoted by i. Each defense firm i has n^i_M production facilities; can utilize n^i_D distribution centers, and can distribute its defense product to the n_R defense demand markets. L^i represents the links of the supply chain network of defense firm i; $i = 1, \ldots, I$, with n_{L^i} elements. By $G = [N, L]$ is denoted the graph consisting of the set of nodes N and the set of links L in Figure 1. The defense supply chain network topology in Figure 1 can be modified/adapted according to the specific defense product under study.

The notation for the model is given in Table 1. The vectors are all column vectors.

The conservation of defense product flow equations are now presented.

The demand for each defense firm's product at each defense demand market must be satisfied by the defense product flows from the defense firm to the defense demand market, as follows: For each defense firm i: $i = 1, \ldots, I$:

$$\sum_{p \in P^i_k} x_p = d_{ik}, \quad k = 1, \ldots, n_R. \tag{1}$$

Furthermore, the the defense product path flows must be nonnegative; where, for each defense firm i; $i = 1, \ldots, I$:

$$x_p \geq 0, \quad \forall p \in P^i. \tag{2}$$

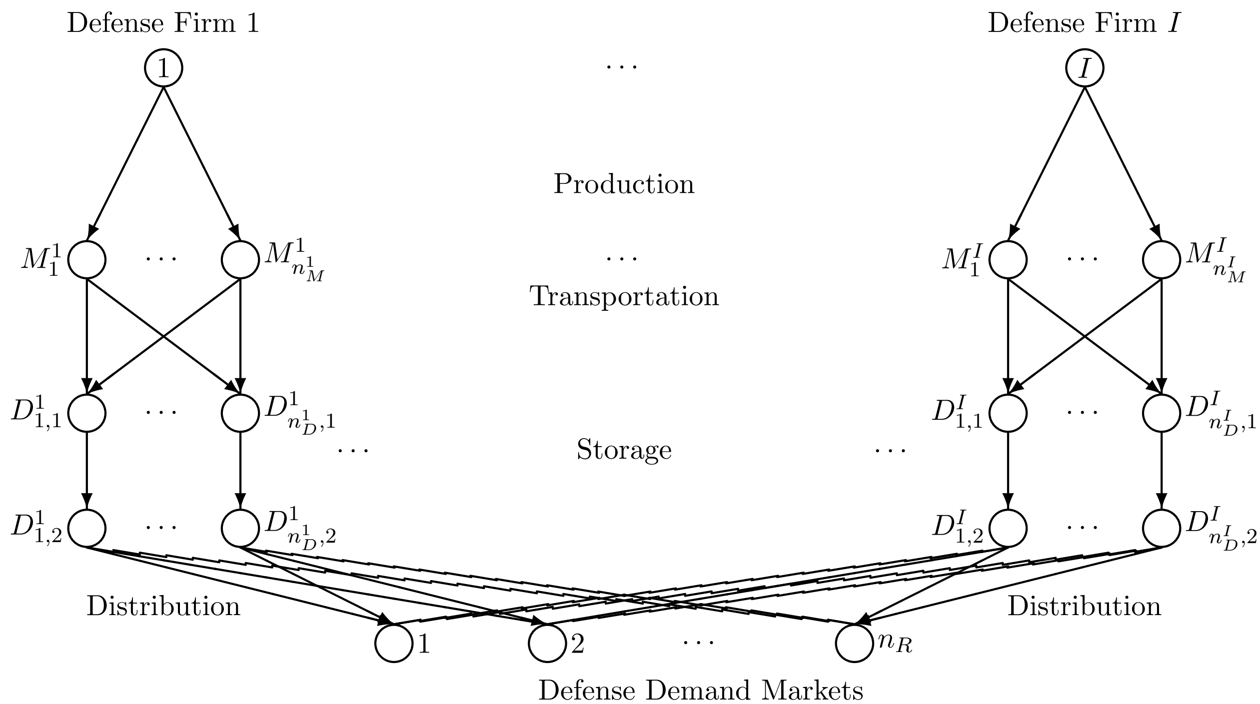

Figure 1: The Defense Critical Supply Chain Network Topology

The link product flows of each defense firm i; $i = 1, \dots, I$, must satisfy the following equations:

$$f_a = \sum_{p \in P^i} x_p \delta_{ap}, \quad \forall a \in L^i, \tag{3}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Note that (3) guarantees that the flow of a defense firm's product on a link is equal to the sum of that defense product's flows on paths that contain that link.

As in Nagurney (2021a, b, c), the product output on each link is a linear function of the labor input, where

$$f_a = \alpha_a l_a, \quad \forall a \in L^i, \quad i = 1, \dots, I. \tag{4}$$

The greater the value of α_a, the more productive the labor on the link. Some economic background on such a construct can be found in Mishra (2007).

We also consider the following constraints on labor, since shortages of skilled labor is a big issue in defense critical supply chains: For each defense firm i; $i = 1, \dots, I$:

$$l_a \leq \bar{l}_a, \quad \forall a \in L^i. \tag{5}$$

The utility function of defense firm i, U^i; $i = 1, \dots, I$, is the profit, consisting of the difference between its revenue and its total costs, the wages paid out, and the weighted total risk:

$$U^i = \sum_{k=1}^{n_R} \rho_{ik}(d) d_{ik} - \sum_{a \in L^i} \hat{c}_a(f) - \sum_{a \in L^i} w_a l_a - \beta_i \sum_{a \in L^i} r_a(f). \tag{6a}$$

Table 1: Notation for the Defense Critical Supply Chain Game Theory Model

Notation	**Definition**
P^i_k	the set of paths in defense firm i's supply chain network ending at defense demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$.
P^i	the set of n_{P^i} paths of defense firm i; $i = 1, \ldots, I$.
P	the set of n_P paths in the defense supply chain network economy.
$x_p; p \in P^i_k$	the nonnegative flow of the defense product of firm i on path p originating at defense firm node i and ending at defense demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$. Defense firm i's defense product path flows are grouped into the vector $x^i \in R^{n_{P^i}}_+$. The defense firms' defense product path flows are grouped into the vector $x \in R^{n_P}_+$.
f_a	the nonnegative flow of the defense product on link a, $\forall a \in L$. The defense product link flows are grouped into the vector $f \in R^{n_L}_+$.
l_a	the labor on link a denoted in person hours, $\forall a \in L$.
α_a	positive factor relating input of labor to output of defense product flow on link a, $\forall a \in L$.
$\bar{l}_a$	the upper bound on the availability of labor on link a, $\forall a \in L$.
d_{ik}	the demand for the defense product of defense firm i at defense demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$. The $\{d_{ik}\}$ elements of defense firm i are grouped into the vector $d^i \in R^{n_R}_+$ and all the defense product demands are grouped into the vector $d \in R^{In_R}_+$.
$\hat{c}_a(f)$	the total operational cost associated with link a, $\forall a \in L$.
$r_a(f)$	the risk function associated with link a, $\forall a \in L$.
β_i	the nonnegative weight applied to the evaluation of the total risk by defense firm i; $i = 1, \ldots, I$. We group all these weights into the vector β.
w_a	the cost (wage) of a unit of labor on link a, $\forall a \in L$.
$\rho_{ik}(d)$	the demand price function for the defense product of defense firm i at defense demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$.

The first expression after the equal sign in (6a) is the revenue of defense firm i. The second expression in (6a) is the total operational costs for the supply chain network L^i of defense firm i; the third expression is the total payout in terms of wages to laborers of defense firm i, and the last term in (6a) is the weighted total risk of defense firm i. The utility functions U_i; $i = 1, \ldots, I$, are assumed to be concave, with the demand price functions being monotone decreasing and continuously differentiable and the total link cost functions being convex and also continuously differentiable with the same assumptions made for the risk functions.

Each defense firm i; $i = 1, \ldots, I$, hence, seeks to solve the following optimization problem:

$$\text{Maximize} \quad \sum_{k=1}^{n_R} \rho_{ik}(d) d_{ik} - \sum_{a \in L^i} \hat{c}_a(f) - \sum_{a \in L^i} w_a l_a - \beta_i \sum_{a \in L^i} r_a(f), \tag{6b}$$

subject to: (1) – (5).

We now demonstrate that the objective function of each firm i; $i = 1, \ldots, I$, can be expressed in path flow variables only. We proceed as follows. In view of (2) and (3), can redefine the total operational cost link functions as: $\tilde{c}_a(x) \equiv \hat{c}_a(f)$, $\forall a \in L$; the demand price functions as: $\tilde{\rho}_{ik}(x) \equiv \rho_{ik}(d)$, $\forall i$, $\forall k$, and the risk functions $\tilde{r}_a(x) \equiv r_a(f)$, $\forall a \in L$. As noted in Nagurney (2021a,b), it follows from (3) and (4), that $l_a = \frac{\sum_{p \in P} x_p \delta_{ap}}{\alpha_a}$, for all $a \in L$.

Hence, one can redefine the utility functions $\tilde{U}^i(x) \equiv U^i$; $i = 1 \ldots, I$, and group the utilities of all the defense firms into an I-dimensional vector $\tilde{U}$, where

$$\tilde{U} = \tilde{U}(x). \tag{7}$$

The optimization problem faced by defense firm i; $i = 1, \ldots, I$, can be expressed as:

$$\text{Maximize} \quad \tilde{U}^i(x) = \sum_{k=1}^{n_R} \tilde{\rho}_{ik}(x) \sum_{p \in P_k^i} x_p - \sum_{a \in L^i} \tilde{c}_a(x) - \sum_{a \in L^i} \frac{w_a}{\alpha_a} \sum_{p \in P^i} x_p \delta_{ap} - \beta_i \sum_{a \in L^i} \tilde{r}_a(x), \tag{8}$$

subject to the nonnegativity constraints (1) and the re-expressing of constraints in (5) as:

$$\frac{\sum_{p \in P^i} x_p \delta_{ap}}{\alpha_a} \leq \bar{l}_a, \quad \forall a \in L^i. \tag{9}$$

2.1 Governing Equilibrium Conditions and Variational Inequality Formulations

The governing equilibrium conditions are now stated, along with alternative variational inequality formulations.

2.1.1 Nash Equilibrium Conditions and Variational Inequality Formulations

The feasible set K_i for defense firm i is defined as: $K_i \equiv \{x^i | x^i \in R_+^{n_{P^i}}, \frac{\sum_{p \in P^i} x_p \delta_{ap}}{\alpha_a} \leq \bar{l}_a, \forall a \in L^i\}$, for $i = 1, \ldots, I$, with $K \equiv \prod_{i=1}^I K_i$. Clearly, K is a convex set.

Since the defense firms are utility-maximizers, they compete noncooperatively until the following Defense Supply Chain Nash Equilibrium is attained.

Definition 1: Defense Supply Chain Network Nash Equilibrium

A defense product path flow pattern $x^* \in K$ is a Defense Supply Chain Network Nash Equilibrium if for each defense firm i; $i = 1, \ldots, I$:

$$\tilde{U}^i(x^{i*}, \hat{x}^{i*}) \geq \tilde{U}^i(x^i, \hat{x}^{i*}), \quad \forall x^i \in K_i, \tag{10}$$

where $\hat{x}^{i*} \equiv (x^{1*}, \ldots, x^{i-1*}, x^{i+1*}, \ldots, x^{I*})$.

Conditions (10) state that a Defense Supply Chain Nash Equilibrium is achieved if no defense firm can improve upon its utility unilaterally.

It follows from the classical theory of Nash equilibria and variational inequalities that, under the imposed assumptions on the total cost, the demand price, and the risk functions, (cf. Gabay and Moulin (1980) and Nagurney (1999)), the solution to the above Defense Supply Chain Nash Equilibrium problem (see Nash (1950, 1951)) coincides with the solution of the variational inequality problem: Determine $x^* \in K$, such that

$$-\sum_{i=1}^{I}\langle \nabla_{x^i}\tilde{U}^i(x^*), x^i - x^{i*}\rangle \geq 0, \quad \forall x \in K, \tag{11}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the corresponding Euclidean space (here, of dimension n_P), and $\nabla_{x^i}\tilde{U}^i(x)$ is the gradient of $\tilde{U}^i(x)$ with respect to x^i.

Existence of a solution to variational inequality (11) is guaranteed since the feasible set K is compact and the utility functions are continuously differentiable, under our imposed assumptions (cf. Kinderlehrer and Stampacchia (1980)).

An alternative variational inequality to (11) is now provided over a simpler feasible set, following the arguments in Nagurney (2021b). The alternative variational inequality is over the nonnegative orthant and will suggest an elegant computational procedure, based on the underlying dynamics as the defense firms adjust their defense product flows over time, with signals provided by Lagrange multipliers associated with the labor link bounds, until a stationary point; equivalently, an equilibrium point satisfying the variational inequality is achieved. We associate Lagrange multipliers λ_a with the constraint (9) for each link $a \in L$ and group the Lagrange multipliers for each defense firm i's supply chain network L^i into the vector λ^i. All such vectors for the defense firms are then grouped into the vector $\lambda \in R_+^{n_L}$. Also, we introduce the feasible sets: $K_i^1 \equiv \{(x^i, \lambda^i) | (x^i, \lambda^i) \in R_+^{n_{P^i} + n_{L^i}}\}$; $i = 1, \ldots, I$, and $K^1 \equiv \prod_{i=1}^{I} K_i^1$.

Theorem 1: Alternative Variational Inequality Formulation of the Defense Supply Chain Nash Equilibrium

The Defense Supply Chain Network Nash Equilibrium satisfying Definition 1 is equivalent to the solution of the variational inequality: determine the vector of equilibrium defense product path flows and the vector of optimal Lagrange multipliers, $(x^*, \lambda^*) \in K^1$, such that:

$$\sum_{i=1}^{I}\sum_{k=1}^{n_R}\sum_{p \in P_k^i}\left[\frac{\partial \tilde{C}_p(x^*)}{\partial x_p} + \beta_i \frac{\partial \tilde{R}_p(x^*)}{\partial x_p} + \sum_{a \in L^i}\frac{\lambda_a^*}{\alpha_a}\delta_{ap} + \sum_{a \in L^i}\frac{w_a}{\alpha_a}\delta_{ap} - \tilde{\rho}_{ik}(x^*) - \sum_{l=1}^{n_R}\frac{\partial \tilde{\rho}_{il}(x^*)}{\partial x_p}\sum_{q \in P_l^i} x_q^*\right] \times [x_p - x_p^*]$$

$$+\sum_{a \in L}\left[\bar{l}_a - \frac{\sum_{p \in P} x_p^* \delta_{ap}}{\alpha_a}\right] \times [\lambda_a - \lambda_a^*] \geq 0, \quad \forall (x, \lambda) \in K^1, \tag{12}$$

where

$$\frac{\partial \tilde{C}_p(x)}{\partial x_p} \equiv \sum_{a \in L^i} \sum_{b \in L^i} \frac{\partial \hat{c}_b(f)}{\partial f_a} \delta_{ap}, \quad \forall p \in P^i, \tag{13}$$

$$\frac{\partial \tilde{R}_p(x)}{\partial x_p} \equiv \sum_{a \in L^i} \sum_{b \in L^i} \frac{\partial r_b(f)}{\partial f_a} \delta_{ap}, \quad \forall p \in P^i. \tag{14}$$

Proof: See proof of Theorem 1 in Nagurney (2021c).

Variational inequality (12) is now put into standard form (cf. Nagurney (1999)), VI$(F, \mathcal{K})$, where one seeks to determine a vector $X^* \in \mathcal{K} \subset R^{\mathcal{N}}$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{15}$$

where F is a given continuous function from $\mathcal{K}$ to $R^{\mathcal{N}}$, $\mathcal{K}$ is a given closed, convex set, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{N}$-dimensional Euclidean space.

In order to put the variational inequality (12) into the form in (15), we let $\mathcal{N} \equiv n_P + n_L$; $X \equiv (x, \lambda)$ and $F(X) \equiv (F^1(X), F^2(X))$, where the p-th component of $F^1(X) \equiv \frac{\partial \tilde{C}_p(x)}{\partial x_p} + \beta_i \frac{\partial \tilde{R}_p(x)}{\partial x_p} + \sum_{a \in L^i} \frac{\lambda_a}{\alpha_a} \delta_{ap} + \sum_{a \in L^i} \frac{w_a}{\alpha_a} \delta_{ap} - \tilde{\rho}_{ik}(x) - \sum_{l=1}^{n_R} \frac{\partial \tilde{\rho}_{il}(x)}{\partial x_p} \sum_{q \in P^i_l} x_q$ and the a-th component of $F^2(X) \equiv \bar{l}_a - \frac{\sum_{p \in P} x_p \delta_{ap}}{\alpha_a}$.

2.2 Dynamics and Algorithm

It is interesting and valuable to also discuss the underlying dynamics as the defense firms adjust their defense product outputs over time and the Lagrange multipliers associated with the link labor bounds also evolve over time. For this purpose, we can apply the theory of projected dynamical systems (cf. Dupuis and Nagurney (993) and Nagurney and Zhang (1996)). We recall the projection operator $\Pi_{\mathcal{K}}(X, v)$:

$$\Pi_{\mathcal{K}}(X, v) = \lim_{\delta \to 0} \frac{(P_{\mathcal{K}}(X + \delta v) - X)}{\delta}, \tag{16}$$

with $P_{\mathcal{K}}$ being the classical projection operator (see Nagurney (1999)). The ordinary differential equation of interest is then:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X^0 \in \mathcal{K}. \tag{17}$$

We know from Theorem 1.23 in Nagurney(1999) that a stationary point X^* of the projected dynamical system (17), which, by definition, satisfies:

$$0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)), \tag{18}$$

coincides with the solution of variational inequality (15).

Specifically, in the context of the model, the rate of change of the defense product flow at a point in time on a path p depends on: $-\frac{\partial \tilde{C}_p(x)}{\partial x_p} - \beta_i \frac{\partial \tilde{R}_p(x)}{\partial x_p} - \sum_{a \in L^i} \frac{\lambda_a}{\alpha_a}\delta_{ap} - \sum_{a \in L^i} \frac{w_a}{\alpha_a}\delta_{ap} + \tilde{\rho}_{ik}(x) + \sum_{l=1}^{n_R} \frac{\partial \tilde{\rho}_{il}(x)}{\partial x_p} \sum_{q \in P_l^i} x_q$ at that point in time, whereas the rate of change of the Lagrange multiplier on a link a depends on $-\bar{l}_a + \frac{\sum_{p \in P} x_p \delta_{ap}}{\alpha_a}$ at the point in time. If the marginal revenue associated with a path of a firm's supply chain network exceeds the marginal costs plus the weighted marginal risk on the path, then the defense product flow will increase; if not, it will decrease, provided that it does not become negative. The projection operator $\Pi_{\mathcal{K}}$ guarantees that the evolution of the product path flows and of the Lagrange multipliers always lies within the feasible set $\mathcal{K}$; in other words, they always remain nonnegative. A plethora of dynamic supply chain network models, including multitiered ones (but without labor), can be found in the book by Nagurney (2006). Nagurney and Ermagun (2022) used the modified projection method of Korpelevich (1977), whereas in this paper, the Euler Method is used.

Observe that (17) represents a continuous-time adjustment process. However, for computational purposes, a discrete-time algorithm that can be easily implemented is needed. For the solution of the model, we propose the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993), with its statement being as follows.

The Euler Method

Initialize with $X^0 \in \mathcal{K}$ and set $\tau = 0$. Compute:

$$X^{\tau+1} = P_{\mathcal{K}}(X^{\tau} - a_{\tau} F(X^{\tau})), \tag{19}$$

where: $\sum_{\tau=0}^{\infty} a_{\tau} = \infty$, $a_{\tau} > 0$, $a_{\tau} \to \infty$, as $\tau \to \infty$.

As mentioned earlier, the feasible set $\mathcal{K}$ for the variational inequality (12) (see also (15)) for the defense supply chain network model is the nonnegative orthant, and, hence, the resolution of the algorithmic scheme in (19) yields closed form expressions for the defense product path flows and for the Lagrange multipliers as stated below.

Explicit Formulae for the Defense Product Path Flows at an Iteration

At iteration $\tau + 1$, one computes the following for each path $p; p \in P_k^i, \forall i, k$:

$$x_p^{\tau+1} = \max\{0, x_a^{\tau} - a_{\tau}(\frac{\partial \tilde{C}_p(x^{\tau})}{\partial x_p} + \beta_i \frac{\partial \tilde{R}_p(x^{\tau})}{\partial x_p} + \sum_{a \in L^i} \frac{\lambda_a^{\tau}}{\alpha_a}\delta_{ap} + \sum_{a \in L^i} \frac{w_a}{\alpha_a}\delta_{ap} - \tilde{\rho}_{ik}(x^{\tau}) - \sum_{l=1}^{n_R} \frac{\partial \tilde{\rho}_{il}(x^{\tau})}{\partial x_p} \sum_{q \in P_l^i} x_q^{\tau})\}; \tag{20}$$

Explicit Formulae for the Lagrange Multipliers at an Iteration

At iteration $\tau + 1$, one computes the following for each Lagrange multiplier $a \in L$:

$$\lambda_a^{\tau+1} = \max\{0, \lambda_a^{\tau} - a_{\tau}(\bar{l}_a - \frac{\sum_{p \in P} x_p^{\tau} \delta_{ap}}{\alpha_a})\}. \quad (21)$$

We apply this algorithm in Section 4 to defense supply chain network examples for which we report the solutions, along with the network performance / efficiency values and additional information, using also results in Section 3.

3. Defense Supply Chain Network Efficiency / Performance

It is important to recognize that, in matters of defense, a government, in preparing for conflicts and/or in times of war, may need to acquire defense supplies from a country other than its own. Our defense supply chain network model allows for this and, we see that this is happening now as the war by Russia against Ukraine rages. Hence, we believe that an adaptation of the constructs for supply chain network performance / efficiency of Nagurney and Qiang (2009) and of Nagurney and Li (2016) can also be applied for the new model in this paper, with note that the new model, unlike the previous ones in the above citations, includes labor; plus, we also have explicit weighted risk functions, since risk is of high relevance in the defense sector.

3.1 Efficiency/Performance of a Defense Supply Chain Network and Importance Identification of a Network Component

The efficiency/performance of a defense supply chain network, denoted by *efficiency*, $\mathcal{E}$, is defined as:

$$\mathcal{E} = \mathcal{E}(G, \hat{c}, \rho, w, r, \beta, \alpha, \bar{l}) \equiv \sum_{i=1}^{I} \sum_{k=1}^{n_R} \frac{\frac{d_{ik}^*}{\rho_{ik}(d^*)}}{In_R}, \quad (22)$$

with the demands, d^*, and the incurred defense demand market prices in (22), evaluated at the solution to (12). Observe that, given a defense supply chain network economy, and the various parameters and functions, the corresponding multi-firm supply chain network is considered as performing better if, on the average, it can handle higher demands at lower prices. Note that, as can be inferred from variational inequality (12), the defense demand market prices capture the information associated with the operational costs, the wages paid out to labor, as well as the weighted risk.

Following then Nagurney and Qiang (2009) for results therein for supply chains and Nagurney and Li (2016), one can then define the importance of a component g (node, link, or a combination of nodes and links), $I(g)$, which represents the efficiency drop when g is removed from the defense supply chain network, as:

$$I(g) = \frac{\Delta\mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G, \hat{c}, \rho, w, r, \beta, \alpha, \bar{l}) - \mathcal{E}(G - g, \hat{c}, \rho, w, r, \beta, \alpha, \bar{l})}{\mathcal{E}(G, \hat{c}, \rho, w, r, \beta, \alpha, \bar{l})}. \tag{23}$$

One can rank the importance of nodes or links, using (23). This formalism can be quite valuable for those engaged in decision-making and policy-making in the military and defense. Those defense supply chain network components that are of higher importance should be paid greater attention to since a disruption to those components will have a bigger overall impact.

Using the above efficiency / performance measure $\mathcal{E}$ one can also quantify the resilience of the defense supply chain network economy to disruptions in labor as discussed in Nagurney and Ermagun (2022), but in the context of a supply chain network optimization model with labor and not a game theory model that also captures risk.

3.2 Resilience Measure Associated with Labor Disruptions

We can adapt the measure proposed in Nagurney and Ermagun (2022) for the defense supply chain network game theory model. As therein, let $\bar{l}\gamma$ denote the reduction of labor availability with $\gamma \in (0, 1]$ so if $\gamma = .8$ this means that the labor availability associated with the labor constraints is now 80% of the original labor availability as in $\mathcal{E}$.

Resilience Measure Capturing Labor Availability

One can define the resilience measure with respect to labor availability, $\mathcal{R}^{\bar{l}\gamma}$, as

$$\mathcal{R}^{\bar{l}\gamma} \equiv \mathcal{R}^{\bar{l}\gamma}(G, \hat{c}, \rho, \pi, \alpha, \bar{l}) = \frac{\mathcal{E}^{\bar{l}\gamma}}{\mathcal{E}} \times 100\%, \tag{24}$$

with $\mathcal{E}$ as in (22).

The expression (24) quantifies the resilience of the defense supply chain network subject to reduction of labor availability. The closer the value is to 100%, the greater the resilience.

4. Numerical Examples

In this Section, the modeling framework is illustrated through numerical examples. The Euler Method was implemented in FORTRAN and a Linux system used for the computations.The $\{a_\tau\}$ sequence used was: $\{10(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)\}$. The algorithm was initialized with a demand of 40 for each demand market of each firm with the demand equally distributed among the paths of each firm. The initial Lagrange multipliers were set to 0.00. The algorithm was deemed to have converged when the absolute value of each computed variable evaluated at two successive iterations differed by no more than 10^{-7}.

The defense supply chain network economy for the specific defense product, which could correspond, for example, to helmets or protective vests, consists of two defense firms, each of which has two production sites, a single distribution center, and serves two defense demand markets, as depicted in Figure 2.

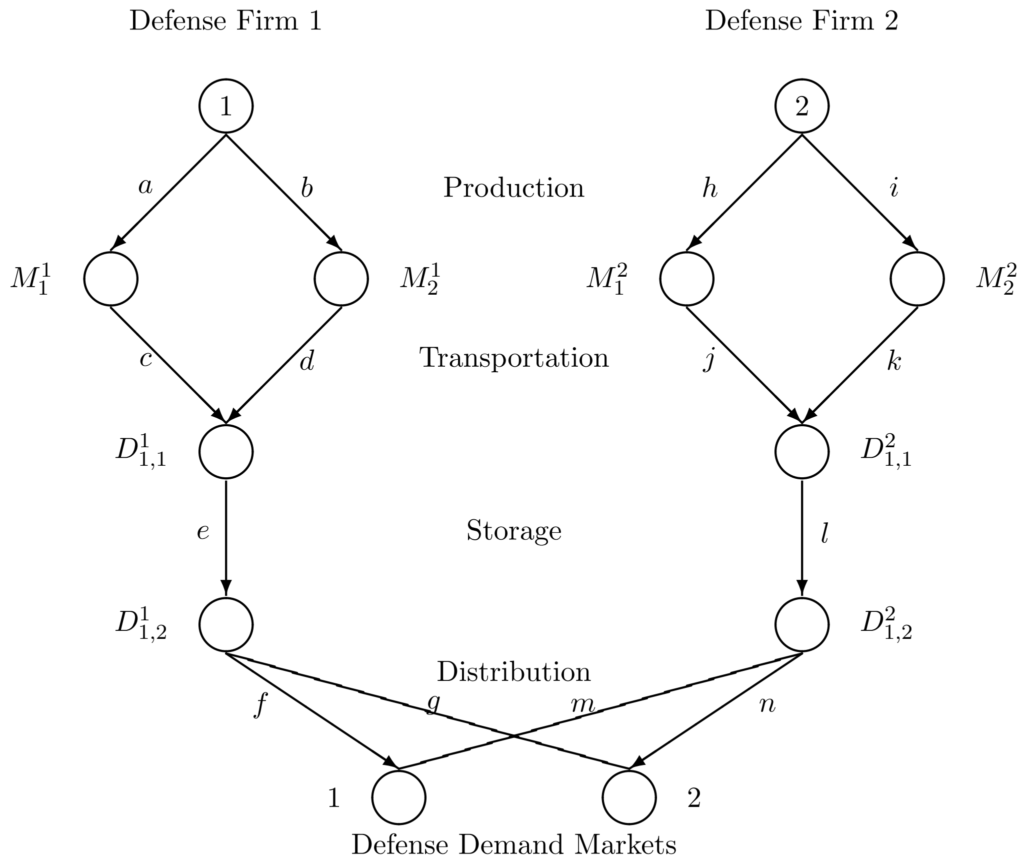

Figure 2: The Supply Chain Network Topology for the Numerical Examples

Example 1 - Baseline

The first example, which serves as the baseline has the following data. Note that, in this example, we assume that the firms are not concerned about risk, so that all the risk functions are identically equal to 0.00.

The total operational cost functions associated with Defense Firm 1's supply chain network links L^1 are:

$$\hat{c}_a(f) = .006f_a^2, \quad \hat{c}_b(f) = .007f_b^2, \quad \hat{c}_c(f) = .01f_c^2, \quad \hat{c}_d(f) = .01f_d^2, \quad \hat{c}_e(f) = .02f_e^2,$$

$$\hat{c}_f(f) = .05f_f^2, \quad \hat{c}_g(f) = .05f_g^2.$$

The total operational costs associated with Defense Firm 2's supply chain network links L^2 are:

$$\hat{c}_h(f) = .0075 f_h^2, \quad \hat{c}_i(f) = .008 f_i^2, \quad \hat{c}_j(f) = .005 f_j^2, \quad \hat{c}_k(f) = .005 f_k^2, \quad \hat{c}_l(f) = .015 f_l^2,$$

$$\hat{c}_m(f) = .1 f_m^2, \quad \hat{c}_n(f) = .1 f_n^2.$$

The hourly labor wages are:

$$w_a = 10, \quad w_b = 10, \quad w_c = 15, \quad w_d = 15, \quad w_e = 20, \quad w_f = 17, \quad w_g = 18,$$

$$w_h = 11, \quad w_i = 22, \quad w_j = 15, \quad w_k = 15, \quad w_l = 18, \quad w_m = 18, \quad w_n = 18.$$

The link labor productivity factors are:

$$\alpha_a = 24, \quad \alpha_b = 25, \quad \alpha_c = 100, \quad \alpha_d = 100, \quad \alpha_e = 50, \quad \alpha_f = 100, \quad \alpha_g = 100,$$

$$\alpha_h = 23, \quad \alpha_i = 24, \quad \alpha_j = 100, \quad \alpha_k = 100, \quad \alpha_l = 70, \quad \alpha_m = 100, \quad \alpha_n = 100.$$

The bounds on labor are:

$$\bar{l}_a = 100, \quad \bar{l}_b = 200, \quad \bar{l}_c = 300, \quad \bar{l}_d = 300, \quad \bar{l}_e = 100, \quad \bar{l}_f = 120, \quad \bar{l}_g = 120,$$

$$\bar{l}_h = 800, \quad \bar{l}_i = 90, \quad \bar{l}_j = 200, \quad \bar{l}_k = 200, \quad \bar{l}_l = 300, \quad \bar{l}_m = 100, \quad \bar{l}_n = 100.$$

The demand price functions of Defense Firm 1 are:

$$\rho_{11}(d) = -.0001 d_{11} - .00005 d_{21} + 600, \quad \rho_{12}(d) = -.0002 d_{12} - .0001 d_{22} + 800.$$

The demand price functions of Defense Firm 2 are:

$$\rho_{21}(d) = -.0003 d_{21} + 700, \quad \rho_{22}(d) = -.0002 d_{22} + 700.$$

The paths are: $p_1 = (a, c, e, f)$, $p_2 = (b, d, e, f)$, $p_3 = (a, c, e, g)$, $p_r = (b, d, e, g)$ for Defense Firm 1 and $p_5 = (h, j, l, m)$, $p_6 = (i, k, l, m)$, $p_7 = (h, j, l, n)$, and $p_8 = (i, k, l, n)$ for Defense Firm 2.

The computed equilibrium defense product path flows are reported in Table 2. The computed equilibrium labor values are reported in Table 3. All the Lagrange multipliers have a value of 0.00 at the equilibrium.

The defense product prices at equilibrium are:

$$\rho_{11} = 599.75, \quad \rho_{12} = 799.10, \quad \rho_{21} = 699.40, \quad \rho_{22} = 699.60,$$

with the equilibrium demands:

$$d_{11}^* = 1506.19, \quad d_{12}^* = 3494.12, \quad d_{21}^* = 1999.04, \quad d_{22}^* = 2001.03.$$

The utility for Defense Firm 1 is: 2,258,772.50 and that for Defense Firm 2 is: 1,649,827.75.

We report the efficiency of this supply chain network, even with all the risk functions set to 0.00. The $\mathcal{E} = 3.15$.

Example 2: Addition of Risk Functions Associated with Production Sites

Example 2 has the same data as that in Example 1, except that now we consider the situation that the production sites are suffering from geopolitical risk and, hence, we have:

$$r_a = f_a^2, \quad r_b(f) = f_b^2, \quad r_h(f) = f_h^2, \quad r_i(f) = f_i^2,$$

with the risk weights of the two firms: $\beta_1 = \beta_2 = 1$.

The computed equilibrium path flows are reported in Table 2, with the computed labor values given in Table 3. All the Lagrange multipliers, again, have a value of 0.00 at the equilibrium. In other words, the respective labor bounds are not reached in Example 2.

The defense product prices at equilibrium are now:

$$\rho_{11} = 599.98, \quad \rho_{12} = 799.83, \quad \rho_{21} = 699.91, \quad \rho_{22} = 699.94,$$

with the equilibrium demands:

$$d_{11}^* = 0.00, \quad d_{12}^* = 690.49, \quad d_{21}^* = 305.50, \quad d_{22}^* = 305.80.$$

The utility for Defense Firm 1 now is: 275,793.59 and that for Defense Firm 2: 213,562.31. One can see that the utilities of both firms have dropped precipitously, in comparison to the utilities that they earned in Example 1, when there was no risk. The efficiency of this defense supply chain network, with risk functions associated with production sites, $\mathcal{E} = .43$. We see that this value is much lower than that in Example 1. We then proceeded to see how resilient this defense supply chain network is with respect to labor disruptions. We calculated $\mathcal{R}^{\bar{l}\gamma}$ for $\gamma = .9, .7, .5, .3, .1$ and found that $\mathcal{R}^{\bar{l}\gamma} = 1$ for all the values of γ noted, except when $\gamma = .1$, where $\mathcal{R}^{\bar{l}.1} = .7$. We can conclude that this defense supply chain network, with the data provided, is quite resilient to labor disruptions.

In Table 4, we report the efficiency of the defense supply chain network for Example 2 when a link g is removed, along with the importance $I(g)$, for $g = a, \ldots, n$. Table 4 provides interesting

Equilibrium Product Path Flows	Ex. 1	Ex. 2
$x^*_{p_1}$	703.17	0.00
$x^*_{p_2}$	803.02	0.00
$x^*_{p_3}$	1696.82	345.41
$x^*_{p_4}$	1797.30	345.08
$x^*_{p_5}$	919.52	152.84
$x^*_{p_6}$	1079.51	152.66
$x^*_{p_7}$	920.51	152.99
$x^*_{p_8}$	1080.52	152.81

Table 2: Equilibrium Defense Product Path Flows for Examples 1 and 2

Equilibrium Link Labor Values	Ex. 1	Ex. 2
l^*_a	100.00	14.39
l^*_b	104.01	13.80
l^*_c	24.00	3.45
l^*_d	26.00	3.45
l^*_e	100.00	13.81
l^*_f	15.06	0.00
l^*_g	34.94	6.90
l^*_h	80.00	13.30
l^*_i	90.00	12.73
l^*_j	18.40	3.06
l^*_k	21.60	3.05
l^*_l	57.14	8.73
l^*_m	19.99	3.05
l^*_n	20.01	3.96

Table 3: Equilibrium Link Labor Values for Examples 1 and 2

results. Overall, one can see that the supply chain network of Defense Firm 2 is more important than that of Defense Firm 1 to this defense supply chain network economy and cognizant governments should make note of this. Indeed, five of the seven links of Defense Firm 2's supply chain network have positive values in terms of their importance. Furthermore, Defense Firm 2's link l, which corresponds to a storage link, has the highest importance value; therefore, every effort should be expended to preserve its functionality. Also, the production link i of Defense Firm 2 merits maintenance and care as do the transportation links j and k. Finally, link m, a distribution link to Defense Demand Market 2, is also of importance. As for the supply chain network of Defense Firm 1, link e, which is a storage link, has the highest value in terms of importance for Defense Firm 1 and, interestingly, its production site associated with link a is of the lowest importance. We emphasize that not only the absolute values in terms of importance of supply chain network components are relevant but also their relative values.

g	$\mathcal{E}(G-g)$	$I(g)$
a	2.43	-4.43
b	.90	-1.08
c	.89	-1.08
d	.89	-1.08
e	.77	-.79
f	1.01	-1.39
g	.99	-1.30
h	.99	-1.30
i	.34	.21
j	.34	.21
k	.34	.21
l	.22	.50
m	.46	-.06
n	.42	.03

Table 4: Efficiency of the Defense Supply Chain Network for Example 2 when Link g Is Removed and the Importance $I(g)$

5. Summary, Conclusions, and Suggestions for Future Research

In this paper, a defense critical supply chain network game theory model was introduced, which includes labor and associated constraints, as well as risk, since current world events have heightened the importance of both risk management and well as resilience of supply chain networks to disruptions, including those associated with labor, which have been significant in the COVID-19 pandemic. The methodological framework for the modeling, analysis, and computations, made use of both variational inequality theory and the theory of projected dynamical systems.

We proposed a noncooperative game theory model consisting of defense firms seeking to supply defense products, that are substitutable to demand markets, which can be associated with different governments that are not antagonistic to one another. The labor constraints are bounds on hours of labor available on the supply chain network links, which are: production, transportation, storage, and distribution links. The utility function of each firm captures revenue as well as weighted risk and the governing equilibrium concept is that of a Nash Equilibrium. Under appropriate assumptions on the utility functions, we provide alternative variational inequality formulations of the governing equilibrium conditions. In addition, a dynamic model is constructed, whose stationary points coincide with the set of solutions to the variational inequality with variables consisting of defense product path flows and Lagrange multipliers associated with the labor constraints. An algorithm, the Euler Method, as the proposed for the time-discretization of the continuous-time trajectories and used in the solution of the numerical examples.

In addition, a network efficiency / performance measure is proposed for the defense supply chain network economy, which can then be applied to quantify the importance of supply chain network components, and then rank them. A resilience measure is also constructed to assess the impacts of disruptions to labor availability.

In order to illustrate the defense supply chain network modeling framework, numerical examples are solved with input and output data reported. The information regarding the defense supply chain network economy, made possible with the tools in the paper, can be useful for decision-makers and policy-makers in governments that are concerned about defense.

It would be interesting, for future research, to investigate the supply chain network efficiency under different kind of labor constraints (see also Nagurney (2021c)) and also under different productivity levels (Nagurney and Ermagun (2022)). It would also be worthwhile to include additional tiers in the supply chain network to include, specifically, suppliers and their behavior, along with labor, and to address issues of supply chain network efficiency and resilience in the defense sector.

Acknowledgments

This paper is dedicated to Ukrainians fighting for the freedom of their country and to the memory of all those who have lost their lives and who are suffering in Russia's war against Ukraine.

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