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Cybersecurity Investments with Nonlinear Budget Constraints and Conservation Laws: Variational Equilibrium, Marginal Expected Utilities, and Lagrange Multipliers

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Abstract

In this paper, we propose a new cybersecurity investment supply chain game theory model, assuming that the demands for the product are known and fixed and, hence, the conservation law of each demand market is fulfilled. The model is a Generalized Nash equilibrium model with nonlinear budget constraints for which we define the variational equilibrium, which provides us with a variational inequality formulation. We construct an equivalent formulation, enabling the analysis of the influence of the conservation laws and the importance of the associated Lagrange multipliers. We find that the marginal expected transaction utility of each retailer depends on this Lagrange multiplier and its sign. Finally, numerical examples with reported equilibrium product flows, cybersecurity investment levels, and Lagrange multipliers, along with individual firm vulnerability and network vulnerability, illustrate the obtained results.

Keywords: cybersecurity; investments; supply chains; conservation laws; game theory; Generalized Nash equilibrium; variational inequalities; Lagrange multipliers

1. Introduction

Supply chains have become increasingly complex as well as global and are now highly dependent on information technology to enhance effectiveness as well as efficiency and to support communications and coordination among the network of suppliers, manufacturers, distributors, and even freight service providers. At the same time, information technology, if not properly secured, can increase the vulnerability of supply chains to cyberattacks. Many examples exist of cyber attacks infiltrating supply chains with a vivid example consisting of the major US retailer Target cyber breach in which attackers entered the system via a third party vendor, an HVAC subcontractor, with an estimated 40 million payment cards stolen in late 2013 and upwards of 70 million other personal records compromised (see [21]). Not only did Target incur financial damages but also reputational costs. Other highly publicized examples have included breaches at the retailer Home Depot, the Sony media company, and the financial services firm JP Morgan Chase. Energy companies as well as healthcare organizations as well as defense companies have also been subject to cyberattacks (cf.[32] and[33]). In addition, the Internet of Things (IoT) has expanded the possible entry points for cyberattacks ([3]).

Of course, cyberattacks are not exclusively a US phenomenon. According to Verizon's 2016 Data Breach Investigations Report, there were 2,260 confirmed data breaches in the previous year at organizations in 82 countries. Numerous other breaches, affecting small and medium-size businesses, have gone unreported and unanalyzed (cf.[40]). In order to illustrate the scope of the negative impacts associated with cybercrime, it has been estimated that the world economy sustained $445 billion in losses from cyberattacks in 2014 (see[2]).

Numerous companies and organizations have now realized that investing in cybersecurity is an imperative. Furthermore, because of the interconnectivity through supply chains and even financial networks, the decisions of an organization in terms of cybersecurity investments can affect the cybersecurity of others. For example, according to Kaspersky Lab, a multinational gang of cybercriminals, known as "Carbanak," infiltrated more than 100 banks across 30 countries and extracted as much as one billion dollars over a period of roughly two years ([23]). Gartner ([27]) and Market Research ([25]) report that organizations in the US are spending $15 billion for security for communications and information systems. Hence, research in cybersecurity investment is garnering increased attention with one of the first research studies on the topic being that of Gordon and Loeb (see[19]).

In this paper we consider a recently studied cybersecurity investment supply chain game theory model consisting of retailers and consumers at demand markets with each retailer being faced with a nonlinear budget constraint on his security investments (see[30] and[8]). We present an alternative to this model in which the demand for the product at each demand market is known and fixed and, hence, the conservation law of each demand market must be fulfilled. The reason for introducing such a satisfaction of the demands at the demand markets is because there are numerous products in which demand is inelastic as in the case, for example, of infant formula, certain medicines, etc.

The supply chain game theory model with cybersecurity investments in the case of fixed, that is, inelastic, demands, unlike the models of[30] and[8], is characterized by a feasible set such that the strategy of a given retailer is affected by the strategies of the other retailers since the product can come from any (or all) of them. Hence, the governing concept is no longer a Nash equilibrium (cf.[35],[36]) but, rather, is a Generalized Nash equilibrium (see, e.g.,[41] and[14]). Recall that, in classical Nash equilibrium problems, the strategies of the players, that is, the decision-makers in the noncooperative game, affect the utility functions of the other players, but the feasible set of each player depends only on his/her strategies. It is worth mentioning that it was Rosen ([37]) who, in his seminal paper, studied a class of GNE problems. In[11] the authors show that the Rosen's class of GNE problems can be solved by finding a solution of a variational inequality. Moreover, the variational solution of a GNE problem with shared constraints has been derived in a general Hilbert space in[13].

In this paper, we make use of a *variational equilibrium* (cf.[12] and[22]), which is a special kind of GNE. The variational equilibrium allows for a variational inequality formulation of the Generalized Nash equilibrium model. Notably, according to[24] and the references therein, the Lagrange multipliers associated with the shared (that is, the common) constraints are the same for all players in the game, which allows for an elegant economic interpretation. In our model, the demand constraints faced by the retailers are the shared ones, and we then fully investigate these and other relevant Lagrange multipliers in this paper.

We note that in the papers [30] and [8] the governing Nash equilibrium conditions are formulated in terms of a variational inequality and an analysis of the dual problem and its associated Lagrange multipliers is performed. In particular, in this paper, the influence of the conservation laws is analyzed and the importance of the associated Lagrange multipliers highlighted. The marginal expected transaction utility for each retailer depends on this Lagrange multiplier and its sign. For other papers on cybersecurity models see also [9], [32], [33], [38], whereas for other studies on the Lagrange theory and its application to variational models we refer to [5], [6], [7], [16], [17], [18], and [39]. For recent research on Generalized Nash equilibrium models in disaster relief supply chains and in commercial supply chains, respectively, see [29] and [34].

In the paper [8] an analysis of the marginal expected cybersecurity investment utilities and their stability is performed and, hence, this paper adds to the literature on the study of marginal expected utilities, with a focus on both supply chains and cybersecurity investments, but in the more challenging setting of Generalized Nash equilibrium.

This paper is organized as follows. In Section 2 we present the model, along with such concepts and firm and network vulnerability, define the variational equilibrium, and provide the variational inequality formulation. In Section 3 we construct an equivalent formulation by means of the Lagrange multipliers associated with the constraints and the conservation law which define the feasible set. Then we prove the existence of the Lagrange multipliers associated with the equality and inequality constraints by applying the Karush-Kuhn-Tucker conditions (see Theorem 3.1). In Section 4 we analyze the marginal expected transaction utilities and we find that they depend on the Lagrange multipliers and their signs. In Section 5 we present detailed numerical examples which emphasize the importance of the Lagrange multipliers and of the inelastic demands in order to maximize the expected utilities. Finally, in Section 6, we present the conclusions and the projects for future research.

2. The Model

The supply chain network, consisting of retailers and consumers at demand markets, is depicted in Figure 1. Each retailer i; $i = 1, \ldots, m$, can transact with demand market j; $j = 1, \ldots, n$, with Q_{ij} denoting the product transaction from i to j. We intend to study the cybersecurity by introducing for each retailer i; $i = 1, \ldots, m$, his cybersecurity or, simply, security, level s_i; $i = 1, \ldots, m$. We group the product transactions for retailer i; $i = 1, \ldots, m$, into the n-dimensional vector Q_i and then we group all such retailer transaction vectors into the mn-dimensional vector Q. The security levels of the retailers are grouped into the m-dimensional vector s.

Then, the cybersecurity level in the supply chain network is the average security and is denoted by $\bar{s}$, where $\bar{s} = \sum_{i=1}^{m} \frac{s_i}{m}$. Also, as in ([32]), a retailer's vulnerability $v_i = 1 - s_i$; $i = 1, \ldots, m$, and the network vulnerability $\bar{v} = 1 - \bar{s}$.

The retailers seek to maximize their individual expected utilities, consisting of expected profits, and compete in a noncooperative game in terms of strategies consisting of their respective product transactions and security levels.

The demand at each demand market j, d_j, is assumed to be fixed and known, in contrast to the models

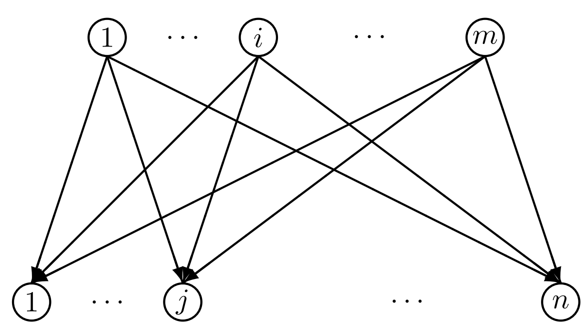

Fig. 1. The Bipartite Structure of the Supply Chain Network Game Theory Model

in[8],[30], and[32]. The demand d_j must satisfy the following conservation law:

$$d_j = \sum_{i=1}^{m} Q_{ij}, \quad j = 1, \ldots, n. \tag{1}$$

The product transactions have to satisfy capacity constraints and must be nonnegative, so that we have the following conditions:

$$0 \leq Q_{ij} \leq \bar{Q}_{ij}, \text{ with } \sum_{i=1}^{m} \overline{Q}_{ij} > d_j \quad i = 1, \ldots, m; j = 1, \ldots, n. \tag{2}$$

The cybersecurity level of each retailer i must satisfy the following constraint:

$$0 \leq s_i \leq u_{s_i}, \quad i = 1, \ldots, m, \tag{3}$$

where $u_{s_i} < 1$ for all i; $i = 1, \ldots, m$. The larger the value of s_i, the higher the security level, with perfect security reflected in a value of 1. However, since, as noted in[30], we do not expect perfect security to be attainable, we have $u_{s_i} < 1$; $i = 1, \ldots, m$. If $s_i = 0$ this means that retailer i has no security.

The demand price of the product at demand market j, $\rho_j(d, s)$; $j = 1, \ldots, n$, is a function of the vector of demands and the network security. We can expect consumers to be willing to pay more for higher network security. In view of the conservation of flow equations above, we can define $\hat{\rho}_j(Q, s) \equiv \rho_j(d, s)$; $j = 1, \ldots, n$. We assume that the demand price functions are continuously differentiable **and concave**.

There is an investment cost function h_i; $i = 1, \ldots, m$, associated with achieving a security level s_i with the function assumed to be increasing, continuously differentiable and convex. For a given retailer i, $h_i(0) = 0$ denotes an entirely insecure retailer and $h_i(1) = \infty$ is the investment cost associated with complete security for the retailer. An example of an $h_i(s_i)$ function that satisfies these properties and that is utilized here (see also[30]) is

$$h_i(s_i) = \alpha_i \left(\frac{1}{\sqrt{(1 - s_i)}} - 1 \right) \text{ with } \alpha_i > 0.$$

The term α_i enables distinct retailers to have different investment cost functions based on their size and needs. Such functions have been introduced by [38] and also utilized by [32]. However, in those models, there are no cybersecurity budget constraints and the cybersecurity investment cost functions only appear in the objective functions of the decision-makers.

In the model with nonlinear budget constraints as in [30] each retailer is faced with a limited budget for cybersecurity investment. Hence, the following nonlinear budget constraints must be satisfied:

$$\alpha_i \left(\frac{1}{\sqrt{(1-s_i)}} - 1 \right) \leq B_i; \quad i = 1, \ldots, m, \tag{4}$$

that is, each retailer can't exceed his allocated cybersecurity budget.

The profit f_i of retailer i; $i = 1, \ldots, m$ (in the absence of a cyberattack and cybersecurity investment), is the difference between his revenue $\sum_{j=1}^{n} \hat{\rho}_j(Q, s) Q_{ij}$ and his costs associated, respectively, with production and transportation: $c_i \sum_{j=1}^{n} Q_{ij} + \sum_{j=1}^{n} c_{ij}(Q_{ij})$, that is,

$$f_i(Q, s) = \sum_{j=1}^{n} \hat{\rho}_j(Q, s) Q_{ij} - c_i \sum_{j=1}^{n} Q_{ij} - \sum_{j=1}^{n} c_{ij}(Q_{ij}), \tag{5}$$

where $c_{ij}(Q_{ij})$ are convex functions.

If there is a successful cyberattack on a retailer i; $i = 1, \ldots, m$, retailer i incurs an expected financial damage given by

$$D_i p_i,$$

where D_i, the damage incurred by retailer i, takes on a positive value, and p_i is the probability of a successful cyberattack on retailer i, where:

$$p_i = (1 - s_i)(1 - \bar{s}), \quad i = 1, \ldots, m, \tag{6}$$

with the term $(1 - \bar{s})$ denoting the probability of a cyberattack on the supply chain network and the term $(1 - s_i)$ denoting the probability of success of such an attack on retailer i. We assume that such a probability is a given data on the basis of statistical observations.

Each retailer i; $i = 1, \ldots, m$, hence, seeks to maximize his expected utility, $E(U_i)$, corresponding to his expected profit given by:

$$E(U_i) = (1 - p_i) f_i(Q, s) + p_i (f_i(Q, s) - D_i) - h_i(s_i) = f_i(Q, s) - p_i D_i - h_i(s_i). \tag{7}$$

Let us remark that, because of the assumptions, $-E(U_i)$ is a convex function.

Let $\mathbb{K}^i$ denote the feasible set corresponding to retailer i, where

$$\mathbb{K}^i \equiv \{(Q_i, s_i) | 0 \leq Q_{ij} \leq \bar{Q}_{ij}, \forall j, \, 0 \leq s_i \leq u_{s_i},$$

and the budget constraint $h_i(s_i) - B_i \leq 0$, holds for $i\}$.

We also define

$$\mathbb{K} \equiv \Big\{(Q, s) \in \mathbb{R}^{mn+m} : -Q_{ij} \leq 0,\ Q_{ij} - \overline{Q}_{ij} \leq 0,\ -s_i \leq 0,$$

$$s_i - u_{s_i} \leq 0,\ h(s_i) - B_i \leq 0,\ i = 1, \ldots, m,\ j = 1, \ldots, n\Big\}.$$

In addition, we define the set of shared constraints $\mathcal{S}$ as follows:

$$\mathcal{S} \equiv \{Q | (1) \text{ holds}\}.$$

We now state the following definition.

Definition 2.1 (A Supply Chain Generalized Nash Equilibrium in Product Transactions and Security Levels) *A product transaction and security level pattern $(Q^*, s^*) \in \mathbb{K}$, $Q^* \in \mathcal{S}$, is said to constitute a supply chain Generalized Nash equilibrium if for each retailer $i; i = 1, \ldots, m$,*

$$E(U_i(Q_i^*, s_i^*, \hat{Q}_i^*, \hat{s}_i^*)) \geq E(U_i(Q_i, s_i, \hat{Q}_i^*, \hat{s}_i^*)), \quad \forall (Q_i, s_i) \in \mathbb{K}^i, \forall Q \in \mathcal{S}, \tag{8}$$

where

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_m^*); \quad \textit{and} \quad \hat{s_i^*} \equiv (s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_m^*).$$

Hence, according to the above definition, a supply chain Generalized Nash equilibrium is established if no retailer can unilaterally improve upon his expected utility (expected profit) by choosing an alternative vector of product transactions and security level, given the product flow and security level decisions of the other retailers and the demand constraints.

We now provide the linkage that allows us to analyze and determine the equilibrium solution via a variational inequality through a variational equilibrium ([22] and [24]).

Definition 2.2 (Variational Equilibrium) *A product transaction and security level pattern (Q^*, s^*) is said to be a variational equilibrium of the above Generalized Nash equilibrium if $(Q^*, s^*) \in \mathbb{K}$, $Q^* \in \mathcal{S}$, is a solution of the variational inequality*

$$-\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} \times \left(Q_{ij} - Q_{ij}^*\right) - \sum_{i=1}^{m} \frac{\partial E(U_i(Q^*, s^*))}{\partial s_i} \times (s_i - s_i^*) \geq 0,$$
$$\forall (Q, s) \in \mathbb{K}, \forall Q \in \mathcal{S}; \tag{9}$$

namely, $(Q^*, s^*) \in \mathbb{K}$, $Q^* \in \mathcal{S}$, is a supply chain Generalized Nash equilibrium product transaction and *security level pattern if and only if it satisfies the variational inequality*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[c_i+\frac{\partial c_{ij}(Q^*_{ij})}{\partial Q_{ij}}-\hat{\rho}_j(Q^*,s^*)-\sum_{k=1}^{n}\frac{\partial\hat{\rho}_k(Q^*,s^*)}{\partial Q_{ij}}\times Q^*_{ik}\right]\times(Q_{ij}-Q^*_{ij})$$
$$+\sum_{i=1}^{m}\left[\frac{\partial h_i(s^*_i)}{\partial s_i}-\left(1-\sum_{k=1}^{m}\frac{s^*_k}{m}+\frac{1-s^*_i}{m}\right)D_i-\sum_{k=1}^{n}\frac{\partial\hat{\rho}_k(Q^*,s^*)}{\partial s_i}\times Q^*_{ik}\right]$$
$$\times(s_i-s^*_i)\geq 0,\quad \forall(Q,s)\in\mathbb{K},\forall Q\in\mathcal{S}. \tag{10}$$

For convenience, we define now the feasible set $\mathcal{K}$ where $\mathcal{K}\equiv\mathbb{K}\cap\mathcal{S}$.

Problem (10) admits a solution since the classical existence theorem, which requires that the set $\mathcal{K}$ is closed, convex, and bounded and the function entering the variational inequality is continuous, is satisfied (see also[26]).

3. Equivalent Formulation of the Variational Inequality

The aim of this section is to find an alternative formulation of the variational inequality (9) for the cybersecurity supply chain game theory model with nonlinear budget constraints and conservation laws by means of the Lagrange multipliers associated with the constraints defining the feasible set $\mathcal{K}$. To this end, we remark that $\mathcal{K}$ can be rewritten in the following way:

$$\mathcal{K} = \Big\{(Q,s)\in\mathbb{R}^{mn+m}: -Q_{ij}\leq 0,\ Q_{ij}-\overline{Q}_{ij}\leq 0,\ -s_i\leq 0,\ s_i-u_{s_i}\leq 0,$$
$$h_i(s_i)-B_i\leq 0,\ \sum_{i=1}^{m}Q_{ij}=d_j,\ i=1,\dots,m,\ j=1,\dots,n\Big\}, \tag{11}$$

and that variational inequality (9) can be equivalently rewritten as a minimization problem. Indeed, by setting:

$$V(Q,s)=-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial E(U_i(Q^*,s^*))}{\partial Q_{ij}}\left(Q_{ij}-Q^*_{ij}\right)-\sum_{i=1}^{m}\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i}\left(s_i-s^*_i\right),$$

we have:

$$V(Q,s)\geq 0 \text{ in } \mathcal{K} \text{ and } \min_{\mathcal{K}}V(Q,s)=V(Q^*,s^*)=0. \tag{12}$$

Then, we can consider the following Lagrange function:

$$\begin{aligned}\mathcal{L}(Q,s,\lambda^1,\lambda^2,\mu^1,\mu^2,\lambda,\gamma) &= V(Q,s)+\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda^1_{ij}(-Q_{ij})\\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda^2_{ij}(Q_{ij}-\overline{Q}_{ij})+\sum_{i=1}^{m}\mu^1_i(-s_i)\\ &+ \sum_{i=1}^{m}\mu^2_i(s_i-u_{s_i})+\sum_{i=1}^{m}\lambda_i(h_i(s_i)-B_i)\\ &+ \sum_{j=1}^{n}\gamma_j\left(\sum_{i=1}^{m}Q_{ij}-d_j\right),\end{aligned} \tag{13}$$

where $(Q,s)\in\mathbb{R}^{mn+m}$, λ^1, $\lambda^2\in\mathbb{R}^{mn}_{+}$, $\mu^1,\mu^2\in\mathbb{R}^m_+$, $\lambda\in\mathbb{R}^m_+$, $\gamma\in\mathbb{R}^n$.

It is worth mentioning that Lagrange function (13) is different from the one considered in [10].

Hence, we are able to prove the following result, which is interesting in itself, namely, using the Mangasarian Fromowitz constraint qualification condition, if (Q^*,s^*) is a solution of variational inequality (9), we are able to prove that KKT conditions (14) hold and vice versa from KKT conditions (14) variational inequality (9) follows. Moreover, for the first time, to the best of our knowledge, we show that strong duality (17) holds.

Theorem 3.1 *The Lagrange multipliers which appear in the Lagrange function* (13) *exist and, for all* $i=1,\dots,m$, *and* $j=1,\dots,n$, *the following conditions hold:*

$$\begin{aligned}&\overline{\lambda}^1_{ij}(-Q^*_{ij})=0,\quad \overline{\lambda}^2_{ij}(Q^*_{ij}-\overline{Q}_{ij})=0,\\ &\overline{\mu}^1_i(-s^*_i)=0,\quad \overline{\mu}^2_i(s^*_i-u_{s_i})=0,\quad \overline{\lambda}_i(h_i(s^*_i)-B_i)=0,\end{aligned} \tag{14}$$

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial Q_{ij}}-\overline{\lambda}^1_{ij}+\overline{\lambda}^2_{ij}+\overline{\gamma}_j=0, \tag{15}$$

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i}-\overline{\mu}^1_i+\overline{\mu}^2_i+\overline{\lambda}_i\frac{\partial h_i(s^*_i)}{\partial s_i}=0. \tag{16}$$

Moreover, also the strong duality holds true; namely:

$$V(Q^*,s^*)=\min_{\mathcal{K}}V(Q,s) \tag{17}$$

$$=\max_{\substack{\lambda^1,\,\lambda^2\in\mathbb{R}^{mn}_+,\,\mu^1,\mu^2\in\mathbb{R}^m_+\\ \lambda\in\mathbb{R}^m_+,\,\gamma\in\mathbb{R}^n}}\min_{(Q,s)\in\mathbb{R}^{mn+m}}\mathcal{L}(Q,s,\lambda^1,\lambda^2,\mu^1,\mu^2,\lambda,\gamma).$$

Proof Since the existence of the solution to problem (10) has been guaranteed, by virtue of the presence of equality constraints, we must apply the KKT theorem (see [20], Theorem 5.8) in order to obtain the existence of the Lagrange multipliers.

Let us denote by (Q^*, s^*) the solution to (10) and let us set:

$$g_i'(Q) = (-Q_{ij})_{j=1,\dots,n} \le 0,\ i = 1, \dots, m;$$
$$I_i'(Q^*) = \{j \in \{1, \dots, n\} : Q^*_{ij} = 0\},\ i = 1, \dots, m;$$
$$g_i''(Q) = \left(Q_{ij} - \overline{Q}_{ij}\right)_{j=1,\dots,n} \le 0,\ i = 1, \dots, m;$$
$$I_i''(Q^*) = \{j \in \{1, \dots, n\} : Q^*_{ij} - \overline{Q}_{ij} = 0\},\ i = 1, \dots, m;$$
$$s_i' = -s_i \le 0,\ i = 1, \dots, m \text{ and } J'_{s_i} = \{i \in \{1, \dots, m\} : s_i^* = 0\},$$
$$s_i'' = s_i - u_{s_i} \le 0,\ i = 1, \dots, m \text{ and } J''_{s_i} = \{i \in \{1, \dots, m\} : s_i^* = u_{s_i}\},$$
$$s_i''' = h(s_i) - B_i \le 0,\ i = 1, \dots, m \text{ and } J'''_{s_i} = \{i \in \{1, \dots, m\} : h(s_i^*) = B_i\},$$
$$h_j(Q) = \sum_{i=1}^{m} Q_{ij} - d_j = 0,\ j = 1, \dots, n.$$

We remark that: $I_i'(Q^*) \cap I_i''(Q^*) = \emptyset$. Define also the matrix:

$$Q = \begin{pmatrix} Q_{11} & \dots & Q_{1j} & \dots & Q_{1n} \\ \dots & & & & \\ Q_{i1} & \dots & Q_{ij} & \dots & Q_{in} \\ \dots & & & & \\ Q_{m1} & \dots & Q_{mj} & \dots & Q_{mn} \end{pmatrix}.$$

For the Karush-Kuhn-Tucker theorem under the Mangasarian Fromowitz constraint qualification condition, we must prove that, taking into account that $\nabla g_i'^T(Q^*) = (-1, \dots, -1)$, there exists $Q \in \mathbb{R}^{mn}$ such that $-Q_{ij} < 0$, $i = 1, \dots, m$ and $j \in I_i'(Q^*)$.

Analogously, since $\nabla g_i''^T(Q^*) = (1, \dots, 1)$, we must also prove that there exists $Q \in \mathbb{R}^{mn}$ such that $Q_{ij} < 0$, $i = 1, \dots, m$ and $j \in I_i''(Q^*)$.

Such a Q does exist, because it is enough to choose $Q_{ij} > 0$ when $j \in I_i'(Q^*)$ and $Q_{ij} < 0$ when $j \in I_i''(Q^*)$.

For what concerns the equality constraints $\sum_{i=1}^{m} Q_{ij} - d_j = 0$; $j = 1, \dots, n$, we must prove that the matrix $\nabla h_j(Q^*)$, $j = 1, \dots, n$ is linearly independent and for some vector $Q \in \mathbb{R}^{mn}$ it must be : $\nabla^T h_j(Q^*) Q < 0$, $j = 1, \dots, n$.

We remark that:

$$\nabla h_j^T(Q^*) = \left(\frac{\partial h_j(Q^*)}{\partial Q_{11}}, \dots, \frac{\partial h_j(Q^*)}{\partial Q_{1n}}, \dots, \frac{\partial h_j(Q^*)}{\partial Q_{m1}}, \dots, \frac{\partial h_j(Q^*)}{\partial Q_{mn}}\right).$$

Hence:

$$\begin{aligned} \nabla h_1^T(Q^*) &= (1, 1, \dots, 1, 0, 0, \dots, 0, \dots, 0, 0, \dots, 0) \\ \dots & \\ \nabla h_n^T(Q^*) &= (0, 0, \dots, 0, 0, 0, \dots, 0, \dots, 1, 1, \dots, 1) \end{aligned}$$

and their linear combination with constants $c_1, \ldots, c_n$ is given by:

$$\sum_{j=1}^{n} c_j \nabla h_j^T(Q^*) = (c_1, \ldots, c_1, c_2 \ldots, c_2, \ldots, c_n, \ldots, c_n) .$$

Such a linear combination is equal to zero if and only if all the coefficients c_j; $j = 1, \ldots, n$, are zero.

As a consequence, $\nabla h_j^T(Q^*)$; $j = 1, \ldots, n$, are linearly independent. Now we have to prove that for a vector Q of the same type as before, we get:

$$\begin{aligned} \nabla h_1^T(Q^*)Q &= \sum_{j=1}^{n} Q_{1j} = 0, \\ \ldots & \\ \nabla h_n^T(Q^*)Q &= \sum_{j=1}^{n} Q_{mj} = 0. \end{aligned} \tag{18}$$

We note that $Q_{1j} > 0$ if $j \in I'_{1j}(Q^*)$ and $Q_{1j} < 0$ if $j \in I''_{1j}(Q^*)$. Moreover, all the components Q_{ij} cannot be simultaneously equal to zero; otherwise, the equality constraint $\sum_{i=1}^{m} Q_{ij} = d_j$ would be unsatisfied. At the same time, it cannot be that $Q_{ij} = \overline{Q}_{ij}$, since $\sum_{i=1}^{m} \overline{Q}_{ij} > d_j$. Therefore, some Q_{ij} are arbitrarily positive and some Q_{ij} are arbitrarily negative and we can choose them so that (18) is verified.

Now, we can proceed with s_i; $i = 1, \ldots, m$. We need to find $s^* \in \mathbb{R}^m$ such that:

$$\begin{array}{ll} \nabla s'_i(s_i^*)s_i < 0 & i \in J'_s(s^*) \\ \nabla s''_i(s_i^*)s_i < 0 & i \in J''_s(s^*), \end{array} \quad \text{namely,} \quad \begin{array}{ll} s_i > 0 & i \in J'_s(s^*) \\ s_i < 0 & i \in J''_s(s^*). \end{array} \tag{19}$$

Moreover, we need: $\nabla \left(h_j(s_i^*) - B_i\right) s_i < 0$, $i \in J'''_s(s^*)$. Since $h_j(s_i)$ is an increasing function, then $\max h_j(s_i) = h_j(u_{s_i})$. Hence, it must be that:

$$\frac{\partial h_j(u_{s_i})}{\partial s_i} s_i < 0, \quad i \in J'''_s(s^*).$$

We recall that $h_j(s_i) = (1 - s_i)^{-\frac{1}{2}}$, which implies $\dfrac{\partial h_j(u_{s_i})}{\partial s_i} = \dfrac{1}{2}(1 - u_{s_i})^{-\frac{3}{2}} > 0$ and that $s_i^* = u_{s_i}$ implies $s_i < 0$, then

$$\frac{\partial h_j(u_{s_i})}{\partial s_i} s_i < 0, \quad i \in J'''_s(s^*).$$

Then, the Lagrange multipliers $\overline{\lambda}^1, \overline{\lambda}^2 \in \mathbb{R}_+^{mn}$, $\overline{\mu}^1, \overline{\mu}^2, \overline{\lambda} \in \mathbb{R}_+^m$, $\overline{\gamma} \in \mathbb{R}^n$, do exist and conditions (14), (15), and (16) hold true (see Th. 5.8 in[20]). Since the inequality constraints are linear or convex and the equality constraints are affine linear, the Lagrange function results to be convex on the whole space

$\mathbb{R}^{3mn+2n+3m}$. Then, by virtue of Theorem 3.8, part b in[20], the point (Q^*, s^*) is the minimal solution of the Lagrange function $\mathcal{L}(Q, s, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})$ in the whole space $\mathbb{R}^{mn+n}$.

As a consequence, taking into account (14), we obtain:

$$\min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma}) = \mathcal{L}(Q^*, s^*, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})$$

$$= V(Q^*, s^*) = \min_{\mathcal{K}} V(Q, s),$$

see also Theorem 5.17 in[20] for similar remarks.

Now, we want to prove the strong duality; namely:

$$V(Q^*, s^*) = \min_{\mathcal{K}} V(Q, s) =$$

$$= \max_{\substack{\lambda^1,\, \lambda^2\in\mathbb{R}^{mn}_+,\, \mu^1,\mu^2\in\mathbb{R}^m_+ \\ \lambda\in\mathbb{R}^m_+,\, \gamma\in\mathbb{R}^n}} \min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma).$$

Indeed, for every λ^1, $\lambda^2 \in \mathbb{R}^{mn}_+$, μ^1, $\mu^2 \in \mathbb{R}^m_+$, $\lambda \in \mathbb{R}^m_+$, $\gamma \in \mathbb{R}^n$, we have:

$$\min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma) \leq \mathcal{L}(Q^*, s^*, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma),$$

and

$$\mathcal{L}(Q^*, s^*, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma) \leq \underbrace{V(Q^*, s^*)}_{=0},$$

since in the Lagrange function all the terms except $V(Q^*, s^*)$ are less than or equal to zero.

Moreover,

$$V(Q^*, s^*) = \min_{\mathcal{K}} V(Q, s) = \min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma}).$$

Further, we also have:

$$\max_{\substack{\lambda^1,\, \lambda^2\in\mathbb{R}^{mn}_+,\, \mu^1,\mu^2\in\mathbb{R}^m_+ \\ \lambda\in\mathbb{R}^m_+,\, \gamma\in\mathbb{R}^n}} \min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma) \leq V(Q^*, s^*)$$

$$\leq \min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})$$

$$\leq \max_{\substack{\lambda^1,\, \lambda^2\in\mathbb{R}^{mn}_+,\, \mu^1,\mu^2\in\mathbb{R}^m_+ \\ \lambda\in\mathbb{R}^m_+,\, \gamma\in\mathbb{R}^n}} \min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma),$$

which yields:

$$V(Q^*, s^*) = \max_{\substack{\lambda^1,\, \lambda^2\in\mathbb{R}^{mn}_+,\, \mu^1,\mu^2\in\mathbb{R}^m_+ \\ \lambda\in\mathbb{R}^m_+,\, \gamma\in\mathbb{R}^n}} \min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma),$$

and the assertion is proved.

Conditions (14)–(16) represent an equivalent formulation of variational inequality (9) and it is easy to see that from (15) and (16) the variational inequality (9) follows. Indeed, multiplying (15) by $(Q_{ij}-Q^*_{ij})$ we obtain:

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial Q_{ij}}(Q_{ij}-Q^*_{ij})-\overline{\lambda}^1_{ij}(Q_{ij}-Q^*_{ij})+\overline{\lambda}^2_{ij}(Q_{ij}-Q^*_{ij})-\overline{\gamma}_j(Q_{ij}-Q^*_{ij})=0$$

and, taking into account (14), we have:

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial Q_{ij}}(Q_{ij}-Q^*_{ij})=\overline{\lambda}^1_{ij}Q_{ij}-\overline{\lambda}^2_{ij}(Q_{ij}-\overline{Q}_{ij})+\overline{\gamma}_j(Q_{ij}-Q^*_{ij})\geq 0.$$

Analogously, multiplying (16) by $(s_i-s^*_i)$, we get:

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i}(s_i-s^*_i)-\overline{\mu}^1_i(s_i-s^*_i)+\overline{\mu}^2_i(s_i-s^*_i)+\overline{\lambda}_i\frac{\partial h_i(s^*_i)}{\partial s_i}(s_i-s^*_i)=0.$$

From (14), we have:

$$\overline{\mu}^1_i(-s^*_i)=0,\quad \overline{\mu}^2_i s^*_i=\overline{\mu}^2_i u_{s_i}.$$

Moreover, if $\overline{\lambda}_i>0$, then $h_i(s^*_i)=B_i=\max h_i(s_i)$, but $h_i(s_i)$ is a nondecreasing function; hence, it attains its maximum value at $s^*_i=u_{s_i}$. Therefore, we get:

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i}(s_i-s^*_i)=\overline{\mu}^1_i s_i-\overline{\mu}^2_i(s_i-u_{s_i})-\overline{\lambda}_i\frac{\partial h_i(s^*_i)}{\partial s_i}(s_i-u_{s_i})\geq 0$$

because $h_i(s_i)$ is a nonnegative convex function such that $h_i(0)=0$. Then $h_i(s_i)$ attains the minimum value at 0. Hence, $\frac{\partial h_i(0)}{\partial s_i}\geq 0$ and, since $\frac{\partial h_i(s_i)}{\partial s_i}$ is increasing, it results in:

$$0\leq\frac{\partial h_i(0)}{\partial s_i}\leq\frac{\partial h_i(s_i)}{\partial s_i},\quad \forall 0\leq s_i\leq u_{s_i}.$$

For the above calculations variational inequality (9) easily follows. □

The term $\frac{\partial E(U_i(Q^*,s^*))}{\partial Q_{ij}}$ is called the *marginal expected transaction utility*, $i=1,\dots,m$; $j=1,\dots,n$, and the term $\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i}$ is called the *marginal expected cybersecurity investment utility*, $i=1,\dots,m$. Our aim is to study such marginal expected utilities by means of (14)–(16).

4. Analysis of Marginal Expected Transaction Utilities and of Marginal Expected Cybersecurity Investment Utilities

From (15) we get

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} - \overline{\lambda}_{ij}^1 + \overline{\lambda}_{ij}^2 + \overline{\gamma}_j = 0, \quad i = 1, \ldots, m;\ j = 1, \ldots, n.$$

So, if $0 < Q_{ij}^* < \overline{Q}_{ij}$, then we get (see also (10))

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} = c_i + \frac{\partial c_{ij}(Q_{ij}^*)}{\partial Q_{ij}} - \hat{\rho}_j(Q^*, s^*) - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k}{\partial Q_{ij}} \times Q_{ik}^* + \overline{\gamma}_j = 0, \tag{20}$$

$$i = 1, \ldots, m;\ j = 1, \ldots, n,$$

whereas if $\overline{\lambda}_{ij}^1 > 0$, and, hence, $Q_{ij}^* = 0$, and $\overline{\lambda}_{ij}^2 = 0$, we get

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} = c_i + \frac{\partial c_{ij}(Q_{ij}^*)}{\partial Q_{ij}} - \hat{\rho}_j(Q^*, s^*) - \sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{\partial \hat{\rho}_k}{\partial Q_{ij}} \times Q_{ik}^* = \overline{\lambda}_{ij}^1 + \overline{\gamma}_j, \tag{21}$$

$$i = 1, \ldots, m;\ j = 1, \ldots, n,$$

and if $\overline{\lambda}_{ij}^2 > 0$, and, hence, $Q_{ij}^* = \overline{Q}_{ij}$, and $\overline{\lambda}_{ij}^1 = 0$, we have

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} = c_i + \frac{\partial c_{ij}(Q_{ij}^*)}{\partial Q_{ij}} - \hat{\rho}_j(Q^*, s^*) - \sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{\partial \hat{\rho}_k}{\partial Q_{ij}} \times Q_{ik}^* = -\overline{\lambda}_{ij}^2 + \overline{\gamma}_j, \tag{22}$$

$$i = 1, \ldots, m;\ j = 1, \ldots, n.$$

Now let us analyze the meaning of equalities (20)–(22). From equality (20), which holds when $0 < Q_{ij}^* < \overline{Q}_{ij}$, we see that for retailer i, who transfers the product Q_{ij}^* to the demand market j, the marginal expected transaction utility is $-\overline{\gamma}_j$. We remark that $-\overline{\gamma}_j \in \mathbb{R}$, but its sign depends on the difference between the marginal expected transaction cost $c_i + \dfrac{\partial c_{ij}(Q_{ij}^*)}{\partial Q_{ij}}$ and the marginal expected transaction revenue $\hat{\rho}_j(Q^*, s^*) + \sum\limits_{\substack{k=1 \\ k \neq i}}^{m} \dfrac{\partial \hat{\rho}_k}{\partial Q_{ij}} \times Q_{ik}^*$. Then the positive situation is the one when $\overline{\gamma}_j > 0$ so that the marginal expected transaction revenues exceed the costs.

Equality (21) shows that, when there is no trade between retailer i and demand market j; namely, $\overline{\lambda}_{ij}^1 > 0$ and equality (21) holds, then the marginal expected transaction utility decreases, whereas if $\overline{\lambda}_{ij}^2 > 0$; namely, $Q_{ij}^* = \overline{Q}_{ij}$, then the marginal expected transaction utility increases.

In conclusion, we remark that the Lagrange variables $\overline{\gamma}_j$, $\overline{\lambda}_{ij}^1$, $\overline{\lambda}_{ij}^2$, $i = 1, \ldots, m$; $j = 1, \ldots, n$, give a precise evaluation of the behavior of the market with respect to the supply chain product transactions.

The analysis of marginal expected cybersecurity investment utilities is the same as the one performed in subsection 3.2 in[8] as well as the stability of the marginal expected cybersecurity investment utilities is the same as the one performed in subsection 3.3 in[8], but we report them here for the reader's convenience. From (16) we have:

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i} - \overline{\mu}_i^1 + \overline{\mu}_i^2 + \overline{\lambda}_i \frac{\partial h_i(s^*)}{\partial s_i} = 0, \quad i = 1, \ldots, m. \tag{23}$$

If $0 < s_i^* < u_{s_i}$, then $\overline{\mu}_i^1 = \overline{\mu}_i^2 = 0$ and we have (see also (10))

$$\begin{aligned}&\frac{\partial h_i(s_i^*)}{\partial s_i} + \overline{\lambda}_i \frac{\partial h_i(s_i^*)}{\partial s_i}\\ &= \left(1 - \sum_{k=1}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i + \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*,s^*)}{\partial s_i} \times Q_{ik}^*.\end{aligned} \tag{24}$$

Since $0 < s_i^* < u_{s_i}$, $h(s_i^*)$ cannot be the upper bound B_i; hence, $\overline{\lambda}_i$ is zero and (24) becomes:

$$\frac{\partial h_i(s_i^*)}{\partial s_i} = \left(1 - \sum_{k=1}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i + \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*,s^*)}{\partial s_i} \times Q_{ik}^*. \tag{25}$$

Equality (25) shows that the marginal expected cybersecurity cost is equal to the marginal expected cybersecurity investment revenue plus the term $\left(1 - \sum_{k=1}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i$; namely, the marginal expected cybersecurity investment revenue is equal to $\frac{\partial h_i(s_i^*)}{\partial s_i} - \left(1 - \sum_{k=1}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i$. This is reasonable because $\left(1 - \sum_{k=1}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i$ is the marginal expected damage expense.

If $\overline{\mu}_i^1 > 0$ and, hence, $s_i^* = 0$, and $\overline{\mu}_i^2 = 0$, we get:

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i} = \frac{\partial h_i(0)}{\partial s_i} - \left(1 - \sum_{\substack{k=1\\k\neq i}}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*,s^*)}{\partial s_i} Q_{ik}^* = \overline{\mu}_i^1. \tag{26}$$

In (26) minus the marginal expected cybersecurity investment utility is equal to $\overline{\mu}_i^1$; hence, the marginal expected cybersecurity cost is greater than the marginal expected cybersecurity investment revenue plus the marginal damage expense. Then the marginal expected cybersecurity investment revenue is less than the marginal expected cybersecurity cost minus the marginal damage expense. We note that case (26) can occur if $\frac{\partial h_i(0)}{\partial s_i}$ is strictly positive.

In contrast, if $\overline{\mu}_i^2 > 0$ and, hence, $s_i^* = u_{s_i}$, retailer j has a marginal gain given by $\overline{\mu}_i^2$, because

$$-\frac{\partial E(U_i(Q^*, u_{s_i}))}{\partial s_i} = -\left(1 - \sum_{\substack{k=1\\k\neq i}}^{m} \frac{u_{s_k}}{m} + \frac{1-u_{s_i}}{m}\right) D_i - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*, s^*)}{\partial s_i} \times Q_{ik}^* + \frac{\partial h_i(u_{s_i})}{\partial s_i} + \overline{\lambda}_i \frac{\partial h_i(u_{s_i})}{\partial s_i} = -\overline{\mu}_i^2. \quad (27)$$

We note that $\overline{\lambda}_i$ could also be positive, since, with $s_i^* = u_{s_i}$, $h_i(s_i)$ could reach the upper bound B_i. In (27) minus the marginal expected cybersecurity investment utility is equal to $-\overline{\mu}_i^2$. Hence, the marginal expected cybersecurity cost is less than the marginal expected cybersecurity investment revenue plus the marginal damage expense. Then the marginal expected cybersecurity investment revenue is greater than the marginal expected cybersecurity cost minus the marginal damage expense.

From (27) we see the importance of the Lagrange variables $\overline{\mu}_i^1$, $\overline{\mu}_i^2$ which describe the effects of the marginal expected cybersecurity investment utilities.

Now let us consider the three cases related to the studied marginal expected cybersecurity investment utilities. Each of these cases holds for certain values of the damage D_i. Let us consider the value D_i for which the first case (25) occurs. We see that in this case there is a unique value of D_i for which (25) holds and if we vary such a value, also the value s_i^* in (25) varies. Now let us consider the value D_i for which (26) holds and let us call D_i^* the value of D_i for which we have

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial s_i} = \frac{\partial h_i(0)}{\partial s_i} - \left(1 - \sum_{\substack{k=1\\k\neq i}}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i^* - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*, s^*)}{\partial s_i} Q_{ik}^* = 0.$$

Then for $0 < D_i < D_i^*$ the solution (Q^*, s^*) to variational inequality (9) remains unchanged because (26) still holds for these new values of D_i and the marginal expected cybersecurity investment utility remains negative, but it is increasing with respect to D_i. Analogously, if we consider the value D_i for which (27) holds and call D_i^* the value such that

$$-\frac{\partial E(U_i(Q^*, u_{s_i}))}{\partial s_i} = -\left(1 - \sum_{\substack{k=1\\k\neq i}}^{m} \frac{u_{s_k}}{m} + \frac{1-u_{s_i}}{m}\right) D_i^* - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*, s^*)}{\partial s_i} \times Q_{ik}^* + \frac{\partial h_i(u_{s_i})}{\partial s_i} + \overline{\lambda}_i \frac{\partial h_i(u_{s_i})}{\partial s_i} = 0,$$

we see that for $D_i > D_i^*$ the solution (Q^*, s^*) to (9) remains unchanged because (27) still holds and the marginal expected cybersecurity investment utility remains positive and is increasing with respect to D_i.

5. Numerical Examples

The numerical examples consist of a supply chain network with two retailers and two demand markets as depicted in Fig. 2.

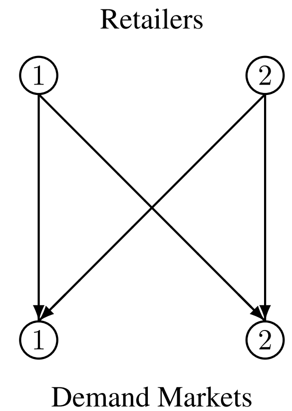

Fig. 2. Network Topology for the Numerical Examples

The examples are inspired by related examples as in [30] and in [8]. Since we want to report all the results for transparency purposes, we have selected the size of problems as reported.
The cost function data are:

$$\begin{array}{rclrcl} c_1 & = & 5, & c_2 & = & 10, \\ c_{11}(Q_{11}) & = & .5Q_{11}^2 + Q_{11}, & c_{12}(Q_{12}) & = & .25Q_{12}^2 + Q_{12}, \\ c_{21}(Q_{21}) & = & .5Q_{21}^2 + Q_{21}, & c_{22}(Q_{22}) & = & .25Q_{22}^2 + Q_{22}. \end{array}$$

The demand price functions are:

$$\rho_1(d, \overline{s}) = -d_1 + .1\frac{s_1 + s_2}{2} + 100, \quad \rho_2(d, \overline{s}) = -.5d_2 + .2\frac{s_1 + s_2}{2} + 200.$$

The damage parameters are: $D_1 = 200$ and $D_2 = 210$ with the investment functions taking the form:

$$h_1(s_1) = \frac{1}{\sqrt{1 - s_1}} - 1, \quad h_2(s_2) = \frac{1}{\sqrt{1 - s_2}} - 1.$$

The damage parameters are in millions of \$US, the expected profits (and revenues) and the costs are also in millions of \$US. The prices are in thousands of dollars and the product transactions are in thousands. The budgets for the two retailers are identical with $B_1 = B_2 = 2.5$ (in millions of \$US). In this case the bounds on the security levels are $u_{s_1} = u_{s_2} = .91$ and the capacities $\overline{Q}_{ij}$ are set to 100 for all i, j.

Keeping the same structure of the network, we have considered five cases with different values of demands:

Case 1: $d_1 = Q_{11} + Q_{21} = 20$ and $d_2 = Q_{12} + Q_{22} = 80$;
Case 2: $d_1 = Q_{11} + Q_{21} = 40$ and $d_2 = Q_{12} + Q_{22} = 190$;
Case 3: no fixed demands;
Case 4: $d_1 = Q_{11} + Q_{21} = 60$ and $d_2 = Q_{12} + Q_{22} = 280$;
Case 5: $d_1 = Q_{11} + Q_{21} = 80$ and $d_2 = Q_{12} + Q_{22} = 380$.

We remark that Case 3 gives the same results as in the example in [8] which is a Nash equilibrium.

For $i = 1$, 2 we obtain:

$$
\begin{aligned}
-\frac{\partial E(U_i(Q,s))}{\partial Q_{i1}} &= 2Q_{i1} + Q_{11} + Q_{21} - .1\frac{s_1+s_2}{2} + c_i - 99,\\
-\frac{\partial E(U_i(Q,s))}{\partial Q_{i2}} &= Q_{i2} + .5Q_{12} + .5Q_{22} - .2\frac{s_1+s_2}{2} + c_i - 199,\\
-\frac{\partial E(U_i(Q,s))}{\partial s_i} &= -\frac{1}{20}Q_{i1} - \frac{1}{10}Q_{i2} - \left(1 - \frac{s_1+s_2}{2} + \frac{1-s_i}{2}\right) D_i\\
&\quad + \frac{1}{2\sqrt{(1-s_i)^3}}.
\end{aligned}
$$

Now, we wish to determine the equilibrium solution, taking into account the different values assumed by λ^1, λ^2, μ^1, μ^2, and λ, and searching, among them, the feasible ones. After some algebraic calculations, we realize that for $i = 1, 2$ and $j = 1, 2$ we get the solution when $\overline{\lambda}^1_{ij} = \overline{\lambda}^2_{ij} = \overline{\mu}^1_i = \overline{\lambda}_i = 0$, and $\overline{\mu}^2_i > 0$. Hence, $s^*_1 = s^*_2 = 0.91$ (which is the maximum value).

In this case, the marginal expected transaction utilities are zero, whereas the marginal expected cybersecurity investment utilities are positive; namely, there is a marginal gain, given by $\overline{\mu}^2_i$, $i = 1, 2$. Solving the system:

$$
\begin{cases}
\dfrac{\partial \mathcal{L}(Q^*, s^*, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})}{\partial Q_{i1}} = 0\\[2ex]
\dfrac{\partial \mathcal{L}(Q^*, s^*, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})}{\partial Q_{i2}} = 0 \qquad i = 1, 2;\\[2ex]
\dfrac{\partial \mathcal{L}(Q^*, s^*, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})}{\partial s_i} = 0
\end{cases}
$$

namely:

$$\begin{cases} 3Q_{11}^* + Q_{21}^* - 0.1\dfrac{s_1^*+s_2^*}{2} + c_1 - 99 - \overline{\lambda}_{11}^1 + \overline{\lambda}_{11}^2 + \overline{\gamma}_1 & = \quad 0 \\ Q_{11}^* + 3Q_{21}^* - 0.1\dfrac{s_1^*+s_2^*}{2} + c_2 - 99 - \overline{\lambda}_{21}^1 + \overline{\lambda}_{21}^2 + \overline{\gamma}_1 & = \quad 0 \\ 1.5Q_{12}^* + .5Q_{22}^* - 0.2\dfrac{s_1^*+s_2^*}{2} + c_1 - 199 - \overline{\lambda}_{12}^1 + \overline{\lambda}_{12}^2 + \overline{\gamma}_2 & = \quad 0 \\ .5Q_{12}^* + 1.5Q_{22}^* - 0.2\dfrac{s_1^*+s_2^*}{2} + c_2 - 199 - \overline{\lambda}_{22}^1 + \overline{\lambda}_{22}^2 + \overline{\gamma}_2 & = \quad 0 \\ -\dfrac{1}{20}Q_{11}^* - \dfrac{1}{10}Q_{12}^* - \dfrac{3-2s_1^*-s_2^*}{2}D_1 + \dfrac{1+\overline{\lambda}_1}{2\sqrt{(1-s_1^*)^3}} - \overline{\mu}_1^1 + \overline{\mu}_1^2 & = \quad 0 \\ -\dfrac{1}{20}Q_{21}^* - \dfrac{1}{10}Q_{22}^* - \dfrac{3-s_1^*-2s_2^*}{2}D_2 + \dfrac{1+\overline{\lambda}_2}{2\sqrt{(1-s_2^*)^3}} - \overline{\mu}_2^1 + \overline{\mu}_2^2 & = \quad 0, \end{cases}$$

and, therefore, assuming for $i = 1,2$, $j = 1,2$, $\overline{\lambda}_{ij}^1 = \overline{\lambda}_{ij}^2 = \overline{\mu}_i^1 = \overline{\lambda}_i = 0$, and $\overline{\mu}_i^2 > 0$; hence, $s_1^* = s_2^* = 0.91$, and $D_1 = 200$ and $D_2 = 210$, we have:

$$\begin{cases} Q_{11}^* + Q_{21}^* & = \quad d_1 \\ 3Q_{11}^* + Q_{21}^* & = \quad 94.091 - \overline{\gamma}_1 \\ Q_{11}^* + 3Q_{21}^* & = \quad 89.091 - \overline{\gamma}_1 \\ Q_{12}^* + Q_{22}^* & = \quad d_2 \\ 1.5Q_{12}^* + .5Q_{22}^* & = \quad 194.182 - \overline{\gamma}_2 \\ .5Q_{12}^* + 1.5Q_{22}^* & = \quad 189.182 - \overline{\gamma}_2 \\ \overline{\mu}_1^2 & = \quad \dfrac{1}{20}Q_{11}^* + \dfrac{1}{10}Q_{12}^* + \dfrac{3-3\times .91}{2}200 - \dfrac{1}{2\sqrt{(1-.91)^3}} \\ \overline{\mu}_2^2 & = \quad \dfrac{1}{20}Q_{21}^* + \dfrac{1}{10}Q_{22}^* + \dfrac{3-3\times .91}{2}210 - \dfrac{1}{2\sqrt{(1-.91)^3}}. \end{cases}$$

The previous system, in the five examined cases, has been solved using Wolfram Alpha and the solutions are summarized in Table 1. In particular, we have reported the flows, the cybersecurity levels, the retailers' vulnerability, the network vulnerability, the Lagrange multipliers associated to the conservation laws and to the constraints on cybersecurity levels, in equilibrium.

	Case 1	**Case 2**	**Case 3**	**Case 4**	**Case 5**
Q_{11}^*	11.25	21.25	24.148	31.25	41.25
Q_{21}^*	8.75	18.75	21.648	28.75	38.75
Q_{12}^*	42.5	97.5	98.341	142.5	192.5
Q_{22}^*	37.5	92.5	93.341	137.5	187.5
s_1^*	.91	.91	.91	.91	.91
s_2^*	.91	.91	.91	.91	.91
v_1	.09	.09	.09	.09	.09
v_2	.09	.09	.09	.09	.09
$\bar{v}$	.09	.09	.09	.09	.09
$\overline{\gamma}_1 = \frac{\partial E(U_1)}{\partial Q_{11}} = \frac{\partial E(U_1)}{\partial Q_{12}}$	51.591	11.591	0	-28.409	- 68.409
$\overline{\gamma}_2 = \frac{\partial E(U_1)}{\partial Q_{12}} = \frac{\partial E(U_1)}{\partial Q_{22}}$	111.682	1.682	0	-88.318	-188.318
$\overline{\mu}_1^2 = \frac{\partial E(U_1)}{\partial s_1}$	13.294	19.294	19.523	24.294	29.794
$\overline{\mu}_2^2 = \frac{\partial E(U_2)}{\partial s_2}$	14.019	20.019	20.248	25.019	30.019

Table 1
Equilibrium solutions

We remark that, since the retailers invest at the upper bound levels of security, both the individual retailers' vulnerability, v_1 and v_2, and that of the network, $\bar{v}$, are low.

	$\rho_1(d^*, s^*)$	$\rho_2(d^*, s^*)$	$E(U_1)$	$E(U_2)$
Case 1	80.091	160.182	2,798.087	5,804.9935
Case 2	60.091	105.182	8,213.3825	7,313.4232
Case 3	54.2954	104.341	8,123.9298	7,156.6968
Case 4	40.091	60.182	3,217.4817	2,455.0132
Case 5	20.091	10.182	-8,732.5083	-9,344.9765

Table 2
Demand prices and expected utilities

Moreover, the demand prices charged by the retailers and the expected utilities of each retailer, in the five cases, are summarized in Table 2.

Comparing the different results, we see that, for some values of the demands, the marginal expected transaction utilities, $\overline{\gamma}_1$ and $\overline{\gamma}_2$, have a positive value; for other values of the demands, $\overline{\gamma}_1$ and $\overline{\gamma}_2$ have negative values, and, when the demand is not fixed at the values above, $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are zero. On the contrary, the marginal cybersecurity investment utilities $\overline{\mu}_1^2$ and $\overline{\mu}_2^2$ are always increasing when the fixed demands increase too and have a small value when the demands are not fixed. Further, for the corresponding values of the demands, the expected utilities, $E(U_i)$; $i = 1, 2$, achieve the maximum value; for the other values of the demands, $E(U_i)$ decrease and, when the demands are not fixed, then $E(U_i)$ assumes a value which is less than the maximum obtained with certain fixed demands. As a conclusion, we can deduce that the problem has an optimal demand which yields optimal expected utilities and a good value of marginal cybersecurity expected utilities, whereas, when the demands are not fixed, we get a value of cybersecurity expected utilities which is not necessarily the optimal one.

Keeping the same structure as the one depicted in Fig. 2, now we study the cybersecurity by introducing the possibility, for each retailer $i = 1, 2$, to have different investment cost functions based on their different sizes and needs.

We assume that the cost functions, the demand price functions, the damage parameters, the budgets for the two retailers, the bounds on the security level, the product transactions capacities are given and are the same as in the previous example, but we suppose now that the investment cost functions are the following:

$$h_1(s_1) = 2\left(\frac{1}{\sqrt{1-s_1}} - 1\right), \quad h_2(s_2) = 3\left(\frac{1}{\sqrt{1-s_2}} - 1\right).$$

Therefore, we are setting $\alpha_1 = 2$ and $\alpha_2 = 3$.

In Table 3 we present the solutions, for the five examined cases, computed using the MatLab program.

In particular we remark that, since the cybersecurity levels are not equal to their upper bounds $(s_i < u_{s_i})$ and the budget constraints are satisfied with equality signs, namely, both retailers use the whole budget, we have: $\bar{\mu}_i^2 = 0$ and $\bar{\lambda}_i \neq 0$.

Comparing the different results, we notice that the product transactions Q_{ij}^* in equilibrium are very similar to the previous ones (when $\alpha_1 = \alpha_2 = 1$) but now the cybersecurity levels are lower, specially when α_i is higher; obviously, in this case, the vulnerability values are bigger.

From Table 4 we also see that the marginal expected cybersecurity investment utilities over the marginal expected cybersecurity costs, the Lagrange multipliers $\bar{\lambda}_1$ and $\bar{\lambda}_2$, are always increasing when the demands increase too.

Furthermore, since $\bar{\lambda}_i > 0$ and $\dfrac{\partial h_i(s^*)}{\partial s_i} > 0$, $i = 1, 2$, for every case, we have that $\dfrac{\partial E(U_i)}{\partial s_i} > 0$.

Moreover, the demand prices charged by the retailers and the expected utilities of each retailer, in the five cases with $\alpha_1 = 2$ and $\alpha_2 = 3$, are summarized in Table 4.

	Case 1	**Case 2**	**Case 3**	**Case 4**	**Case 5**
Q_{11}^*	11.25	21.25	24.1438	31.25	41.25
Q_{21}^*	8.75	18.75	21.6438	28.75	38.75
Q_{12}^*	42.5	97.5	98.3252	142.5	192.5
Q_{22}^*	37.5	92.5	93.3252	137.5	187.5
s_1^*	.8025	.8025	.8025	.8025	.8025
s_2^*	.7025	.7025	.7025	.7025	.7025
v_1	.1975	.1975	.1975	.1975	.1975
v_2	.2975	.2975	.2975	.2975	.2975
$\bar{v}$	.2475	.2475	.2475	.2475	.2475
$\overline{\gamma}_1 = \frac{\partial E(U_1)}{\partial Q_{11}} = \frac{\partial E(U_1)}{\partial Q_{12}}$	51.5752	11.5752	0	-28.4248	- 68.4248
$\overline{\gamma}_2 = \frac{\partial E(U_1)}{\partial Q_{12}} = \frac{\partial E(U_1)}{\partial Q_{22}}$	111.6505	1.6505	0	-88.3495	-188.35
$\overline{\lambda}_1 = \frac{\frac{\partial E(U_1)}{\partial s_1}}{\frac{\partial h_1(s^*)}{\partial s_1}}$	5.5028	6.0295	6.0495	6.4685	6.9514
$\overline{\lambda}_2 = \frac{\frac{\partial E(U_2)}{\partial s_2}}{\frac{\partial h_2(s^*)}{\partial s_2}}$	8.4566	9.1057	9.1303	9.6466	10.2417

Table 3
Equilibrium solutions with $\alpha_1 = 2$ and $\alpha_2 = 3$

	$\rho_1(d^*, s^*)$	$\rho_2(d^*, s^*)$	$E(U_1)$	$E(U_2)$
Case 1	80.0772	160.1545	6,857.8143	6,028.8489
Case 2	60.0772	105.1545	8,202.0839	7,858.6184
Case 3	54.2954	104.341	8,114.7336	7,785.6368
Case 4	40.0772	60.1545	3,204.8084	3,273.8429
Case 5	20.0772	10.1545	-8,746.6946	-8,227.6601

Table 4
Demand prices and expected utilities with $\alpha_1 = 2$ and $\alpha_2 = 3$

6. Conclusions

In this paper, we introduced a cybersecurity investment supply chain game theory model consisting of retailers and consumers at demand markets assuming that the demands for the product at the demand markets are known and fixed and, hence, the conservation law of each demand market is fulfilled. The model also has nonlinear budget constraints. This model is a Generalized Nash equilibrium model since not only are the retailers' expected utility functions dependent on one another's strategies but their feasible sets are as well. We proposed a variational equilibrium which allows us to formulate the governing equilibrium conditions as a variational inequality problem, rather than a quasi-variational inequality. We also studied the dual problem and, specifically, we analyzed the Lagrange multipliers associated with the conservation laws and the expected utilities when the demands change. In particular, we have seen that, for certain values of the fixed demand, we can attain the best expected utilities with respect to the demand. In the future we would like to continue the study of this topic and, in particular, we will take into account uncertainty on the data which leads to a random formulation of the model (see also[4] for an application to the traffic network models).

The results in this paper add to the growing literature of operations research and game theory techniques for cybersecurity modeling and analysis.

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References

[1] Bensoussan A (1974) Points de Nash dans le cas de Fonctionelles Quadratiques et Jeux Differentiels Lineaires a N Personnes. SIAM Journal on Control 12: 460-499.

[2] Center for Strategic and International Studies (2014) Net Losses: Estimating the Global Cost of Cybercrime. Available at: http://www.mcafee.com/us/resources/reports/rp-economic-impact-cybercrime2.pdf/ Accessed 12.04.16.

[3] ComputerWeekly.com (2015) Business disruption cyber attacks set to spur defence plans, says Gartner. Available at: http://www.computerweekly.com/news/2240241129/Business-disruption- cyber-attacks-set-to-spur-defence-plans-says-Gartner/ Accessed 12.04.16.

[4] Daniele P, Giuffré S (2015) Random Variational Inequalities and the Random Traffic Equilibrium Problem. J. Optim. Theory Appl. 167(1): 363-381.

[5] Daniele P, Giuffré S, Lorino M (2016) Functional Inequalities, Regularity and Computation of the Deficit and Surplus Variables in the Financial Equilibrium Problem. Journal of Global Optimization 65(1): 575-596.

[6] Daniele P, Giuffré S, Idone G, Maugeri A (2007) Infinite Dimensional Duality and Applications. Math. Ann. 339: 221-239.

[7] Daniele P, Giuffré S, Maugeri A, Raciti F (2014) Duality Theory and Applications to Unilateral Problems. J. Optim. Theory Appl. 162: 718-734.

[8] Daniele P, Maugeri A, Nagurney A (2017) Cybersecurity Investments with Nonlinear Budget Constraints: Analysis of the Marginal Expected Utilities, in Operations Research, Engineering, and Cyber Security: Trends in Applied Mathematics and Technology, Daras NJ, Rassias MT Eds., Springer Optim. Appl. 113: 117-134.

[9] Daras NJ, Rassias MT (2015) Computation, Cryptography, and Network Security, Springer International Publishing Switzerland.

[10] Facchinei F, Pang JS (2003) Finite-Dimensional Variational Inequalities and Complementarity Problems Volume I, New York, Springer-Verlag.

[11] Facchinei F, Fischer A, Piccialli V (2007) On Generalized Nash Games and Variational Inequalities. Operations Research Letters 35 (2): 159-164.

[12] Facchinei F, Kanzow C (2010) Generalized Nash Equilibrium Problems. Annals of Operations Research 175: 177-211.

[13] Faraci F, Raciti F (2015) On Generalized Nash Equilibrium in Infinite Dimension: the Lagrange Multipliers Approach. Optimization 64 (2): 321-338.

[14] Fischer A, Herrich M, Schonefeld K (2014) Generalized Nash Equilibrium Problems - Recent Advances and Challenges. Pesquisa Operacional 34(3): 521-558.

[15] Gabay D, Moulin H (1980) On the Uniqueness and Stability of Nash Equilibria in Noncooperative Games, in Applied Stochastic Control of Econometrics and Management Science, Eds. Bensoussan A, Kleindorfer P, Tapiero CS, North-Holland, Amsterdam, The Netherlands, pp. 271-294.

[16] Giuffré S, Idone G, Maugeri A (2005) Duality Theory and Optimality Conditions for Generalized Complementarity Problems. Nonlinear Analysis, Theory, Methods and Applications 63 (5-7): e1655-e1664.

[17] Giuffré S, Maugeri A, Puglisi D (2015) Lagrange Multipliers in Elastic-Plastic Torsion Problem for Nonlinear Monotone Operators. J. Differential Equations 259(3): 817-837.

[18] Giuffré S, Maugeri A (2014) A Measure-Type Lagrange Multiplier for the Elastic-Plastic Torsion. Nonlinear Anal. 102: 23-29.

[19] Gordon LA, Loeb MP (2002) The Economics of Information Security Investment. ACM Transactions on Information and System Security 5(4): 438-457.

[20] Jahn J (1994) Introduction to the Theory of Nonlinear Optimization, Berlin, Springer-Verlag.

[21] Kirk, J (2014) Target Contractor Says It Was Victim of Cyberattack. PC World, February 6.

[22] Kulkarni AA, Shanbhag UV (2012) On the Variational Equilibrium as a Refinement of the Generalized Nash Equilibrium. Automatica 48: 45-55.

[23] Lennon, M (2015) Hackers Hit 100 Banks in 'Unprecedented' $1 Billion Cyber Heist: Kaspersky Lab. Security Week, February 15.

[24] Luna JP (2013) Decomposition and Approximation Methods for Variational Inequalities, with Applications to Deterministic and Stochastic Energy Markets. PhD Thesis, Instituto Nacional de Matematica Pura e Aplicada, Rio de Janeiro, Brazil.

[25] Market Research (2013) United States Information Technology Report Q2 2012, April 24.

[26] Maugeri A, Raciti F (2009) On Existence Theorems for Monotone and Nonmonotone Variational Inequalities. Journal of Convex Analysis 16 (3-4), 899-911.

[27] Messmer, E. (2013) Gartner reveals top 10 security myths. Network- World, June 11.

[28] Nagurney A (2015) A Multiproduct Network Economic Model of Cybercrime in Financial Services. Service Science 7(1): 70-81.

[29] Nagurney A, Alvarez Flores E, Soylu C (2016) A Generalized Nash Equilibrium Network Model for Post-Disaster Humanitarian Relief. Transportation Research E 95: 1-18.

[30] Nagurney A, Daniele P, Shukla S (2017) A Supply Chain Network Game Theory Model of Cybersecurity Investments with Nonlinear Budget Constraints. Ann. Oper. Res. 248(1): 405-427.

[31] Nagurney A, Nagurney LS (2015) A Game Theory Model of Cybersecurity Investments with Information Asymmetry. Netnomics 16(1-2): 127-148.

[32] Nagurney A, Nagurney LS, Shukla S (2015) A Supply Chain Game Theory Framework for Cybersecurity Investments Under Network Vulnerability. In Eds. Daras NJ and Rassias MT, Computation, Cryptography, and Network Security, Springer International Publishing Switzerland, pp. 381-398.

[33] Nagurney A, Shukla S (2017) Multifirm Models of Cybersecurity Investment Competition vs. Cooperation and Network Vulnerability. European Journal of Operational Research, 260(2): 588-600.

[34] Nagurney A, Yu M, Besik D (2017) Supply Chain Network Capacity Competition with Outsourcing: A Variational Equilibrium Framework. Journal of Global Optimization, 69(1): 231-254.

[35] Nash JF (1950) Equilibrium Points in n-Person Games. Proceedings of the National Academy of Sciences, USA 36: 48-49.

[36] Nash JF (1951) Noncooperative Games. Annals of Mathematics 54: 286-298.

[37] Rosen JB (1965) Existence and Uniqueness of Equilibrium Points for Concave n-person Games. Econometrica 33: 520-534.

[38] Shetty N, Schwartz G, Felegehazy M, Walrand J (2009) Competitive Cyber-Insurance and Internet Security. Proceedings of the Eighth Workshop on the Economics of Information Security (WEIS 2009), University College London, England, June 24-25.

[39] Toyasaki F, Daniele P, Wakolbinger T (2014) A Variational Inequality Formulation of Equilibrium Models for End-of-Life Products with Nonlinear Constraints. European J. Oper. Res. 236(1): 340-350.

[40] Verizon (2016) Data Breach Investigations Report; available at http://www.verizonenterprise.com/verizon-insights-

lab/dbir/2016/
[41] von Heusinger A (2009) Numerical Methods for the Solution of the Generalized Nash Equilibrium Problem. PhD Dissertation, University of Wurtburg, Germany.