# A Cournot-Nash–Bertrand Game Theory Model of a Service-Oriented Internet with Price and Quality Competition Among Network Transport Providers


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**Abstract:** This paper develops a game theory model of a service-oriented Internet in which profit-maximizing service providers provide substitutable (but not identical) services and compete with the quantities of services in a Cournot-Nash manner, whereas the network transport providers, which transport the services to the users at the demand markets, and are also profit-maximizers, compete with prices in Bertrand fashion and on quality. The consumers respond to the composition of service and network provision through the demand price functions, which are both quantity and quality dependent. We derive the governing equilibrium conditions of the integrated game and show that it satisfies a variational inequality problem. We then describe the underlying dynamics, and provide some qualitative properties, including stability analysis. The proposed algorithmic scheme tracks, in discrete-time, the dynamic evolution of the service volumes, quality levels, and the prices until an approximation of a stationary point (within the desired convergence tolerance) is achieved. Numerical examples demonstrate the modeling and computational framework.



**Key words:** network economics, game theory, oligopolistic competition, service differentiation, quality competition, Cournot-Nash equilibrium, service-oriented Internet, Bertrand competition, variational inequalities, projected dynamical systems

## 1. Introduction

The Internet has transformed the ways in which individuals, groups, and organizations communicate, obtain information, access entertainment, and conduct their economic and social activities. Many users, if not the majority, however, are unaware of the economics underlying the provision of various Internet services. Indeed, as argued in Nagurney et al. (2013), although the technology associated with the existing Internet is rather well-understood, the economics of the associated services have been less studied. Moreover, modeling and computational frameworks that capture the competitive behavior of decision-makers ranging from service providers to network providers are still in their infancy. This may be due, in part, to unawareness of appropriate methodological frameworks.

In this paper, we build on the recent work on game theory frameworks for a service-oriented Internet with the goal of expanding the generality of applicable game theory models that are also computable. By services we mean not only content, such as news, videos, music, etc., but also services associated with, for example, cloud computing. Our work is inspired by that of Zhang et al. (2010) who employed Cournot and Bertrand games to model competition among service providers and among network providers, with the former competing in a Cournot manner, and the latter in a Bertrand manner. The two types of competition were then unified in a Stackelberg game. The authors, however, focused only on a two service provider, two network provider, and two user network configuration along with a linear demand function to enable closed form analytical solutions. In addition, they did not capture the quality of network provision. Altman et al. (2011) emphasized the need for metrics for quality of service and the Internet and also provided an excellent review of game theory models, with a focus on net-neutrality, and noted that many of the models in the existing literature considered only one or two service providers.

In this paper, we focus on the development of a game theory model, in both equilibrium and dynamic settings, that captures competition among service providers and among network (transport of service) providers. The former competition is assumed to be that of Cournot-Nash since the service providers compete in service volumes (quantities) (cf. Cournot (1838)), whereas the latter is that of Bertrand (1838) competition, since the network providers compete in prices, along with quality levels. We note that Nagurney et al. (2013) also developed a competitive Cournot-Nash game theory model for a service-oriented Internet but did not include the profit-maximizing behavior of the network providers. Here, we allow for quality competition associated with network provision, whereas Nagurney et al. (2013) considered exclusively quality competition in service production. Moreover, in this paper, we also describe the underlying dynamics and we integrate non-cooperative behavior (cf. Nash (1950, 1951)) for both the service providers and the network providers.

In our framework, we do not restrict the number of service providers, nor the number of network providers, nor users. Moreover, we allow for service differentiation. Our model also enables the tracking not only of the volume of services provided but also the evolution of the quality levels as well as the prices that the network providers charge. The methodology that we utilize for the integrated network economic model of the service-oriented Internet under Cournot-Nash-Bertrand equilibrium is variational inequality theory (cf. Nagurney (1999)). For its dynamic counterpart, we use projected dynamical systems theory (see Nagurney and Zhang (1996) and Nagurney (2006)).

A notable feature of our modeling approach is that it allows for *composition*, in that users at demand markets have associated demand price functions that reflect how much they are willing to pay for the service and the network provision combination, as a function of service volumes and quality levels. Such an idea is motivated, in part, to provide consumers with more choices (see Wolf et al. (2012)). Consequently, our framework can be used as the foundation for the further disaggregation of decision-making and the inclusion of additional topological constructs, say, in expanding the paths, which may reflect the transport of services at the more detailed level of expanded sequences of links.

Our contributions in the paper fall under *network economics* as well as *computational management science*. Some of the early papers on network economics and the Internet are the contributions of MacKie-Mason and Varian (1995), Varian (1996), Kelly (1997), MacKnight and Bailey (1997), Kausar, Briscoe, and Crowcroft (1999), and Odlyzko (2000). More recent contributions include the works of: Ros and Tuffin (2004), He and Walrand (2005), Shakkottai and Srikant (2006), Shen and Basar (2007), and Neely (2007). Lv and Rouskas (2010), in an interesting paper, focused on the modeling of Internet service providers and the pricing of tiered network services. They provided both models and an algorithm, along with computational results, a contribution that is rare in this stream of literature. However, they assumed that the users are homogeneous, whereas we consider distinct demand price functions associated with the demand markets and the composition of service provider services and network provision.

Dynamic pricing of network use has been explored previously, in particular in the context of cellular access networks, where network resources are limited and congestion can occur easily. For example, Dyaberi et al. (2012) and Ha et al. (2012) have explored the use of incentives and time-dependent pricing to match user needs with operator constraints to reduce congestion. Both of these works use data from real user experiments to derive a price-demand function. Similarly, Varian (2001) provides similar data from a study that solely focused on wired access bandwidth. While these works are limited to one specific network access type and do not consider advanced services, they do provide a useful starting point for obtaining model parameters from real-world users. See also Sen et al. (2012) for a survey on network economics and time-dependent pricing with a goal of relieving congestion management. We note that our transportation cost functions also capture congestion, in addition to the quality of the transport.

We also note the work of Njoroge et al. (2009) who addressed the interconnection between network providers, the endogenous quality choice by network and service providers, and the market coverage in tandem. They studied the competition between service providers in the quality levels of their services and the competition between network providers in their quality levels and prices in a network with two network providers and multiple service providers. Hence, the focus was on duopoly competition. In their framework, consumers and content providers base their selection of ISP on extra features such as speed of access, special add-ons such as spam blocking and virus protection, with these extra features abstracted as *quality*. In addition, they make their selections based on price. Our model is not restricted to duopolistic competition but has the quality variables reflecting extra features as in the above-noted paper.

The paper is organized as follows. In Section 2, we develop the game theory model by explicitly describing the service providers' and the network providers' competitive behavior and interactions with the users at the demand markets and with one another. We demonstrate that the governing integrated Cournot-Nash-Bertrand equilibrium conditions are equivalent to the solution of a variational inequality problem. In Section 3, we then identify the underlying dynamics associated with the time evolution of the service volumes, the quality levels, and the prices charged by the network providers. We show that the dynamics correspond to a projected dynamical system, whose set of stationary points coincides with the set of solutions to the variational inequality problem governing the Cournot-Nash-Bertrand model in Section 2. We also provide some qualitative properties, including stability analysis.

In Section 4, we propose an algorithm, which yields a time-discretization of the continuous-time adjustment processes in service volumes, quality levels, and prices until an approximation of the stationary point (within the desired convergence tolerance) is achieved. We also give convergence results. The algorithm is then applied to compute solutions to several numerical examples, in Section 5, in order to illustrate the modeling and computational framework. We summarize our results and present our conclusions in Section 6.

## 2. The Cournot-Nash-Bertrand Game Theory Model with Price and Quality Competition

In this section, we develop a game theory model of a service-oriented Internet in which we capture the behavior of both the service providers and the network providers. We assume that there are $m$ service providers, with a typical service provider denoted by $i$, $n$ network providers, which provide "transport" of the services to the demand markets, with a typical one denoted by $j$, and $o$ demand markets associated with the users of the services and network provision. A typical demand market is denoted by $k$. The service providers offer multiple different services such as movies for video streaming, music for downloading, news, etc. Users can select among different service offerings (e.g., movie streaming from service provider 1 vs. movie streaming from service provider 2). Different network providers can be used for data communication over the Internet (i.e., "transport") between the service providers and the users. In addition, we explicitly handle the quality level among network providers.

We allow for consumers to differentiate among the services provided by the service providers. It is assumed that the service providers compete under the Cournot-Nash equilibrium concept of non-cooperative behavior and select their service volumes (quantities). The network providers, in turn, compete with prices a la Bertrand and with quality levels. The consumers, in turn, signal their preferences for the services and network provision via the demand price functions associated with the demand markets. The demand price functions are, in general, functions of the service/network provision combinations at all the demand markets as well as the quality levels of network provision, since the focus here is on *composition* and having choices.

The notation for the game theory model is given in Table 1, whereas Figure 1 depicts the network structure of the game theory problem. All vectors here are assumed to be column vectors. An optimal/equilibrium solution is denoted by a "*."

In Section 2.1 we present the behavior of the service providers, along with the Nash-Cournot definition and formulation. In Section 2.2, we then describe the analogues for the network providers, but under Bertrand competition. In Section 2.3, we present the integrated Cournot-Nash-Bertrand equilibrium and derive the variational inequality formulation of the governing equilibrium conditions.

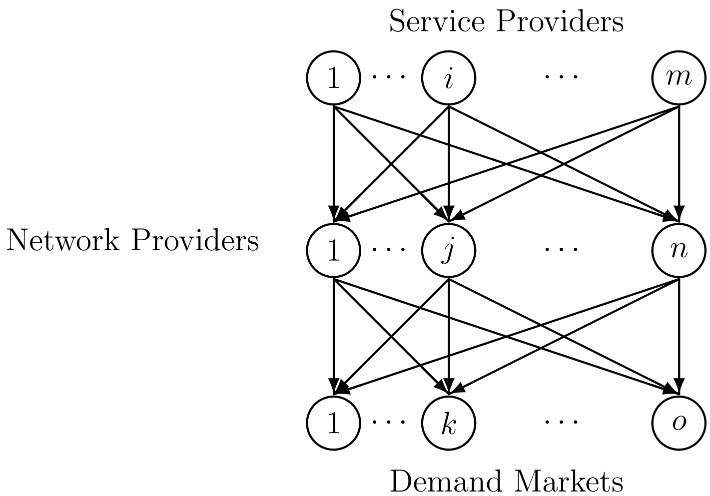

Figure 1: The Network Structure of the Cournot-Nash-Bertrand Model for a Service-Oriented Internet

Table 1: Notation for the Game Theoretic Cournot-Nash-Bertrand Model

| **Notation** | **Definition** |
|---|---|
| $Q_{ijk}$ | the nonnegative service volume from $i$ to $k$ via $j$. We group the $\{Q_{ijk}\}$ elements for all $j$ and $k$ into the vector $Q_i \in R_+^{no}$ and then we group all the vectors $Q_i$ for all $i$ into the vector $Q \in R_+^{mno}$. |
| $s_i$ | the service volume (output) produced by service provider $i$. We group the $\{s_i\}$ elements into the vector $s \in R_+^m$. |
| $q_{ijk}$ | the nonnegative quality level of network provider $j$ transporting service $i$ to $k$. We group the $q_{ijk}$ for all $i$ and $k$ into the vector $q_j \in R_+^{mo}$ and all the vectors $q_j$ for all $j$ into the vector $q \in R_+^{mno}$. |
| $\pi_{ijk}$ | the price charged by network provider $j$ for transporting a unit of service provided by $i$ via $j$ to $k$. We group the $\pi_{ijk}$ for all $i$ and $k$ into the vector $\pi_j \in R_+^m$ and then we group all the vectors $\pi_j$ for all $j$ into the vector $\pi \in R_+^{mno}$. |
| $f_i(s)$ | the total production cost of service provider $i$. |
| $\hat{\rho}_{ijk}(Q, q)$ | the demand price at $k$ associated with service $i$ transported via $j$. |
| $c_{ijk}(Q, q)$ | the total transportation cost associated with delivering service $i$ via $j$ to $k$. |
| $oc_{ijk}(\pi_{ijk})$ | the opportunity cost associated with pricing by network provider $j$ services transported from $i$ to $k$. |

### 2.1 The Behavior of the Service Providers and Their Optimality Conditions

The service providers seek to maximize their individual profits, where the profit function for service provider $i$; $i = 1, \ldots, m$ is given by the expression:

$$\sum_{j=1}^{n}\sum_{k=1}^{o} \hat{\rho}_{ijk}(Q, q^*) Q_{ijk} - f_i(s) - \sum_{j=1}^{n}\sum_{k=1}^{o} \pi^*_{ijk} Q_{ijk} \tag{1}$$

subject to the constraints:

$$s_i = \sum_{j=1}^{n}\sum_{k=1}^{o} Q_{ijk}, \quad i = 1, \ldots, m, \tag{2}$$

$$Q_{ijk} \geq 0, \quad j = 1, \ldots, n; k = 1, \ldots, o. \tag{3}$$

The first term in (1) is the revenue for service provider $i$, the second term is his production cost, and the third term in (1) is the total payout to the network providers for delivering the services to the users at the demand markets. Note that a service provider $i$ controls his service volumes $Q_i$, whereas, as we show in the next subsection, the network providers control the prices charged for the transport of the services from the service providers to the users as well as the quality levels of such transport. Hence, we have $q^*$ and $\pi^*$ in (1).

According to constraint (2), the quantity of the service produced by each service provider is equal to the sum of the amounts of service transported to all the demand markets via all the network providers. Constraint (3) guarantees that the service volumes are nonnegative.

In view of constraint (2), we can define the production cost functions $\hat{f}_i(Q)$; $i = 1, \ldots, m$, as follows:

$$\hat{f}_i(Q) \equiv f_i(s). \tag{4}$$

We assume that the production cost and the demand price functions are continuous and continuously differentiable. We also assume that the production cost functions are convex and that the demand price functions are monotonically decreasing in service volumes but increasing in the quality of network provision.

Therefore, the profit maximization problem for service provider $i$; $i = 1, \ldots, m$, with its profit expression denoted by $U^1_i$, which also represents its utility function, with the superscript 1 reflecting the first (top) tier of decision-makers in Figure 1, can be reexpressed as:

$$\text{Maximize } U^1_i(Q, q^*, \pi^*) = \sum_{j=1}^{n}\sum_{k=1}^{o} \hat{\rho}_{ijk}(Q, q^*) Q_{ijk} - \hat{f}_i(Q) - \sum_{j=1}^{n}\sum_{k=1}^{o} \pi^*_{ijk} Q_{ijk} \tag{5}$$

subject to: $Q_i \in K^{1i}$, where $K^{1i} \equiv \{Q_i | Q_i \geq 0\}$. We also define $K^1 \equiv \prod_{i=1}^{m} K^{1i}$.

Note that the consumers pay the service providers according to the incurred prices $\{\hat{\rho}_{ijk}\}$, whereas the service providers pay the transport network providers according to the equilibrium prices $\{\pi^*_{ijk}\}$ with the former evaluated at the equilibrium service volumes and quality levels (as described further below). The consumers at the demand markets reflect their preferences for the composition of services (combination of service and transport network provision) through the demand price functions.

We assume that the service providers compete according to Cournot-Nash. Indeed, note that the production cost functions (4) capture competition for resources since the production cost of a particular service provider depends not only on his service volume, but also on that of the other service providers. Also, the demand price functions (see Table 1) reveal that users at a demand market care not only about the quality level associated with their specific service/network provision combination but also on that of the other combinations, as well as the service volumes.

In view of (1) - (5), we may write the profit functions of the service providers as functions of the service provision/transportation pattern, that is,

$$U^1 = U^1(Q, q, \pi), \tag{6}$$

where $U^1$ is the $m$-dimensional vector with components: $\{U^1_1, \ldots, U^1_m\}$.

We consider the oligopolistic market mechanism, in which the $m$ service providers supply their services in a non-cooperative fashion, each one trying to maximize its own profit. We seek to determine a nonnegative service volume pattern $Q^*$ for which the $m$ service providers will be in a state of equilibrium as defined below. In particular, Nash (1950, 1951) generalized Cournot's concept of an equilibrium among several players, in what has been come to be called a non-cooperative game.

**Definition 1: Cournot-Nash Equilibrium with Service Differentiation and Network Provision Choices.** *A service volume pattern $Q^* \in K^1$ is said to constitute a Cournot-Nash equilibrium if for each service provider $i$; $i = 1, \ldots, m$:*

$$U^1_i(Q^*_i, \hat{Q}^*_i, q^*, \pi^*) \geq U^1_i(Q_i, \hat{Q}^*_i, q^*, \pi^*), \quad \forall Q_i \in K^{1i}, \tag{7}$$

*where*

$$\hat{Q}^*_i \equiv (Q^*_1, \ldots, Q^*_{i-1}, Q^*_{i+1}, \ldots, Q^*_m). \tag{8}$$

According to (7), a Cournot-Nash equilibrium is established if no service provider can unilaterally improve upon its profits by selecting an alternative vector of service volumes. Alternative variational inequality formulations of the above equilibrium are:

**Theorem 1: Variational Inequality Formulations of Cournot-Nash Equilibrium.** *Assume that for each service provider i the profit function* $U_i^1(Q,q,\pi)$ *is concave with respect to the variables in* $\{Q_i\}$ *and is continuous and continuously differentiable. Then,* $Q^* \in K^1$ *is a Cournot-Nash equilibrium according to Definition 1 if and only if it satisfies the variational inequality*

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\frac{\partial U_i^1(Q^*,q^*,\pi^*)}{\partial Q_{ijk}} \times (Q_{ijk}-Q^*_{ijk}) \geq 0, \quad \forall Q \in K^1, \tag{9}$$

*or, equivalently,* $Q^* \in K^1$ *is a Cournot-Nash equilibrium service volume pattern if and only if it satisfies the variational inequality*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}} + \pi^*_{ijk} - \hat{\rho}_{ijk}(Q^*,q^*) - \sum_{h=1}^{n}\sum_{l=1}^{o}\frac{\partial \hat{\rho}_{ihl}(Q^*,q^*)}{\partial Q_{ijk}} \times Q^*_{ihl}\right] \times (Q_{ijk}-Q^*_{ijk}) \geq 0,$$

$$\forall Q \in K^1. \tag{10}$$

**Proof:** (9) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987).

In order to obtain (10) from (9), we note that:

$$-\frac{\partial U_i^1(Q^*,q^*,\pi^*)}{\partial Q_{ijk}} = \left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}} + \pi^*_{ijk} - \hat{\rho}_{ijk}(Q^*,q^*) - \sum_{h=1}^{n}\sum_{l=1}^{o}\frac{\partial \hat{\rho}_{ihl}(Q^*,q^*)}{\partial Q_{ijk}} \times Q^*_{ihl}\right], \forall i,j,k. \tag{11}$$

Multiplying the expression in (11) by $(Q_{ijk}-Q^*_{ijk})$ and summing the resultant over all $i$, $j$, and $k$ yields (10). □

## 2.2 The Behavior of the Network Providers and Their Optimality Conditions

The network providers also seek to maximize their individual profits. They have as their strategic variables the prices that they charge for the transport of the services and the quality levels of the transport.

We denote the profit function associated with network provider $j$ by $U_j^2$ since this is the second tier of decision-makers (cf. Figure 1). The optimization problem faced by network provider $j$; $j=1,\ldots,n$ is given by

$$\text{Maximize } U_j^2(Q^*,q,\pi) = \sum_{i=1}^{m}\sum_{k=1}^{o}\pi_{ijk}Q^*_{ijk} - \sum_{i=1}^{m}\sum_{k=1}^{o}(c_{ijk}(Q^*,q) + oc_{ijk}(\pi_{ijk})) \tag{12}$$

subject to:

$$\pi_{ijk} \geq 0, \quad i=1,\ldots,m; k=1,\ldots,o, \tag{13}$$

$$q_{ijk} \geq 0, \quad i = 1, \ldots, m; k = 1, \ldots, o. \tag{14}$$

The first term in (12) after the equal sign is the revenue, whereas the second term is the total transportation cost. Note that the total transportation cost is quite general and captures also competition in quality as well as possible congestion associated with transport to the demand markets. The third term in (12) is the opportunity cost, which captures that, if the price charged by the network provider is too high then there is an associated business cost; similarly, if it is too low, since then other costs may not be adequately covered. The opportunity cost may include, for example, possible anticipated regulatory costs, loss of potential revenue if the price charged is too high, etc. We, hence, assume that the costs in (12) are convex, continuous, and continuously differentiable. Observe that the network providers have, as their strategic variables, the prices and quality levels of the transport provided. They do not directly control the volume of services that they transport and, therefore, the use of $Q^*$ in (12).

We group the network provider utility functions, as given in (12), into the vector $U^2$:

$$U^2 = U^2(Q, q, \pi). \tag{15}$$

Let $K^{2j}$ denote the feasible set corresponding to network provider $j$, such that $K^{2j} \equiv \{(q_j, \pi_j) | q_j \geq 0, \pi_j \geq 0\}$ and define $K^2 \equiv \prod_{j=1}^{n} K^{2j}$.

We now define the Bertrand equilibrium that captures the network providers' behavior.

**Definition 2: Bertrand Equilibrium in Transport Prices and Quality.** *A quality level pattern and transport price pattern $(q^*, \pi^*) \in K^2$ is said to constitute a Bertrand equilibrium if for each network provider $j$; $j = 1, \ldots, n$:*

$$U_j^2(Q^*, q_j^*, \hat{q_j^*}, \pi_j^*, \hat{\pi_j^*}) \geq U_j^2(Q^*, q_j, \hat{q_j^*}, \pi_j, \hat{\pi_j^*}), \quad \forall (q_j, \pi_j) \in K^{2j}, \tag{16}$$

*where*

$$\hat{q_j^*} \equiv (q_1^*, \ldots, q_{j-1}^*, q_{j+1}^*, \ldots, q_n^*), \tag{17}$$

$$\hat{\pi_j^*} \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_{j+1}^*, \ldots, \pi_n^*). \tag{18}$$

According to (16), a Bertrand equilibrium is established if no network provider can unilaterally improve upon its profits by selecting an alternative vector of quality levels and transport prices. Alternative variational inequality formulations of the above equilibrium are:

**Theorem 2: Variational Inequality Formulations of Bertrand Equilibrium.** *Assume that for each network provider $j$ the profit function $U_j^2(Q, q, \pi)$ is concave with respect* *to the variables in* $\{q_j\}$ *and in* $\{\pi_j\}$ *and is continuous and continuously differentiable. Then,* $(q^*, \pi^*) \in K^2$ *is a Bertrand equilibrium according to Definition 2 if and only if it satisfies the variational inequality*

$$-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial U_j^2(Q^*, q^*, \pi^*)}{\partial q_{ijk}} \times (q_{ijk} - q_{ijk}^*) - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial U_j^2(Q^*, q^*, \pi^*)}{\partial \pi_{ijk}} \times (\pi_{ijk} - \pi_{ijk}^*) \geq 0,$$

$$\forall (q, \pi) \in K^2, \tag{19}$$

*or, equivalently,* $(q^*, \pi^*) \in K^2$ *is a Bertrand price and quality level equilibrium pattern if and only if it satisfies the variational inequality*

$$\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\sum_{h=1}^{m}\sum_{l=1}^{o}\frac{\partial c_{hjl}(Q^*, q^*)}{\partial q_{ijk}}\right] \times (q_{ijk} - q_{ijk}^*)$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\left[-Q_{ijk}^* + \frac{\partial oc_{ijk}(\pi_{ijk}^*)}{\partial \pi_{ijk}}\right] \times (\pi_{ijk} - \pi_{ijk}^*) \geq 0, \qquad \forall (q, \pi) \in K^2. \tag{20}$$

**Proof:** Similar to the proof of Theorem 1.

### 2.3 The Integrated Cournot-Nash-Bertrand Equilibrium Conditions and Variational Inequality Formulations

We are now ready to present the Cournot-Nash-Bertrand equilibrium conditions. We let $K^3 \equiv K^1 \times K^2$ denote the feasible set for the integrated model. We assume the same assumptions on the functions as in Sections 2.1 and 2.2.

**Definition 3: Cournot-Nash-Bertrand Equilibrium in Service Differentiation, Transport Network Prices, and Quality.** *A service volume, quality level, and transport price pattern* $(Q^*, q^*, \pi^*) \in K^3$ *is a Cournot-Nash-Bertrand equilibrium if it satisfies (7) and (16) simultaneously.*

Given Definition 3, Theorem 3 below is immediate.

**Theorem 3: Variational Inequality Formulations of Cournot-Nash-Bertrand Equilibrium.** *Under the same assumptions as given in Theorems 1 and 2,* $(Q^*, q^*, \pi^*) \in K^3$ *is a Cournot-Nash-Bertrand equilibrium according to Definition 3 if and only if it satisfies the variational inequality:*

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\frac{\partial U_i^1(Q^*, q^*, \pi^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q_{ijk}^*)$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial U_j^2(Q^*,q^*,\pi^*)}{\partial q_{ijk}}\times(q_{ijk}-q_{ijk}^*)-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\frac{\partial U_j^2(Q^*,q^*,\pi^*)}{\partial \pi_{ijk}}\times(\pi_{ijk}-\pi_{ijk}^*)\geq 0,$$
$$\forall (Q,q,\pi)\in K^3, \tag{21}$$

*or, equivalently, the variational inequality problem:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}}+\pi_{ijk}^*-\hat{\rho}_{ijk}(Q^*,q^*)-\sum_{h=1}^{n}\sum_{l=1}^{o}\frac{\partial \hat{\rho}_{ihl}(Q^*,q^*)}{\partial Q_{ijk}}\times Q_{ihl}^*\right]\times(Q_{ijk}-Q_{ijk}^*)$$
$$+\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\sum_{h=1}^{m}\sum_{l=1}^{o}\frac{\partial c_{hjl}(Q^*,q^*)}{\partial q_{ijk}}\right]\times(q_{ijk}-q_{ijk}^*)$$
$$+\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{o}\left[-Q_{ijk}^*+\frac{\partial oc_{ijk}(\pi_{ijk}^*)}{\partial \pi_{ijk}}\right]\times(\pi_{ijk}-\pi_{ijk}^*)\geq 0,\quad \forall (Q,q,\pi)\in K^3. \tag{22}$$

Indeed, note that if we set $(q,\pi)=(q^*,\pi^*)$ and substitute into variational inequality (22), we obtain (10). Similarly, if we let $Q=Q^*$ and substitute into (22) we obtain (20). Hence, the solution of (22) also provides us with the solutions to (10) and (22).

We now put variational inequality (22) into standard form (cf. Nagurney (1999)): determine $X^*\in\mathcal{K}$ where $X$ is a vector in $R^N$, $F(X)$ is a continuous function such that $F(X):X\mapsto\mathcal{K}\subset R^N$, and

$$\langle F(X^*),X-X^*\rangle\geq 0,\quad \forall X\in\mathcal{K}, \tag{23}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in the $N$-dimensional Euclidean space, and $\mathcal{K}$ is closed and convex. We define the vector $X\equiv(Q,q,\pi)$ and $\mathcal{K}\equiv K^3$. Also, here $N=3mno$. The components of $F$ are then given by: for $i=1,\dots,m$; $j=1,\dots,n$; $k=1,\dots,o$:

$$F_{ijk}^1(X)=\frac{\partial \hat{f}_i(Q)}{\partial Q_{ijk}}+\pi_{ijk}-\hat{\rho}_{ijk}(Q,q)-\sum_{h=1}^{n}\sum_{l=1}^{o}\frac{\partial \hat{\rho}_{ihl}(Q,q)}{\partial Q_{ijk}}\times Q_{ihl}, \tag{24}$$

$$F_{ijk}^2(X)=\sum_{h=1}^{m}\sum_{l=1}^{o}\frac{\partial c_{hjl}(Q,q)}{\partial q_{ijk}}, \tag{25}$$

$$F_{ijk}^3(X)=-Q_{ijk}+\frac{\partial oc_{ijk}(\pi_{ijk})}{\partial \pi_{ijk}}. \tag{26}$$

Hence, (22) can be put into standard form (23).

### 2.3.1 An Illustrative Example and a Variant

We now present a simple example for illustrative purposes. Please refer to Figure 2. The problem consists of a single service provider, a single network provider, and a single demand market.

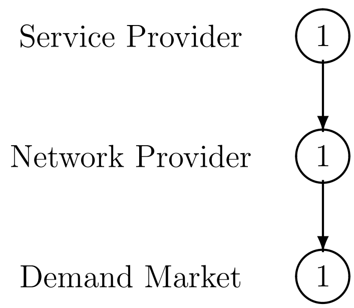


Figure 2: Network Topology for an Illustrative Example

The functions are as follows. The production cost function is:

$$\hat{f}_1(Q_{111}) = Q_{111}^2 + Q_{111}.$$

The demand price function is:

$$\hat{\rho}_{111}(Q_{111}, q_{111}) = -2Q_{111} + q_{111} + 78,$$

and the total transportation cost function is:

$$c_{111}(Q_{111}, q_{111}) = (q_{111} - 1)^2,$$

with an opportunity cost of:

$$oc_{111}(\pi_{111}) = \pi_{111}^2.$$

Hence, according to (24):

$$F_{111}^1(X) = 2Q_{111} + 1 + \pi_{111} + 2Q_{111} - q_{111} - 78 + 2Q_{111}$$

$$= 6Q_{111} + \pi_{111} - q_{111} - 77,$$

whereas, according to (25):

$$F_{111}^2(X) = 2q_{111} - 2$$

and, according to (26):

$$F_{111}^3(X) = -Q_{111} + 2\pi_{111}.$$

We will assume that the Cournot-Nash-Bertrand equilibrium solution $X^* = (Q^*, q^*, \pi^*)$ in this example lies in the interior, so we can then explicitly solve for $X^*$ in (22), with notice that $\mathcal{K}$ is the nonnegative orthant $R^3_+$ as follows. Hence, we can set $F^1_{111}(X^*) = 0$, $F^2_{111}(X^*) = 0$, and $F^3_{111}(X^*) = 0$. Since $F^2_{111}(X^*) = 0$, this means that

$$2q^*_{111} - 2 = 0$$

so that $q^*_{111} = 1$. Also, since $F^3_{111}(X^*) = 0$. we know that

$$Q^*_{111} = 2\pi^*_{111}.$$

Noting that $F^1_{111}(X^*) = 0$, simplifies to

$$6Q^*_{111} + \pi^*_{111} - q^*_{111} - 77 = 0,$$

which, with the above substitutions, yields:

$$13\pi^*_{111} = 78$$

or

$$\pi^*_{111} = 6.$$

Hence, $X^* = (12, 1, 6)$ and the profit of service provider 1, $U^1_1 = 432$, and that of network provider 1, $U^2_1 = 36$.

We now construct a variant of the above example. All the data remain the same except that we change the transportation cost function of the network provider, which is now:

$$c_{111}(Q_{111}, q_{111}) = (q_{111} - 1)^2 + Q_{111}q_{111}$$

so that the service volume explicitly appears now in the last term.

The new equilibrium solution is now: $X^* = (11.82, 0, 5.91)$. Note that the quality level has dropped to zero. The profit $U^1_1(X^*)$ for the service provider is now 421.14, whereas that for the network provider, $U^2_1(X^*)$, is now 33.10. Hence, both providers now have lower profits than in the original example.

### 2.3.2 Another Illustrative Example

We now present another example. There are now two service providers, a single network provider, and a single demand market, as depicted in Figure 3.

The data are as follows.

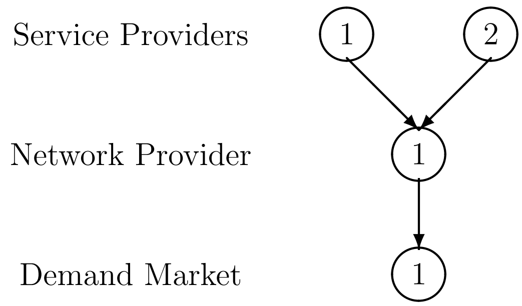


Figure 3: Network Topology for Another Illustrative Example

The production cost functions are:

$$\hat{f}_1(Q) = Q_{111}^2 + Q_{111}, \quad \hat{f}_2(Q) = 2Q_{211}^2 + Q_{211}.$$

The demand price functions are:

$$\hat{\rho}_{111}(Q,q) = -Q_{111} - .5Q_{211} + .5q_{111} + 100, \quad \hat{\rho}_{211}(Q,q) = -Q_{211} - .5Q_{111} + .5q_{211} + 200.$$

The transportation cost functions are:

$$\hat{c}_{111}(Q,q) = .5(q_{111} - 20)^2, \quad \hat{c}_{211}(Q,q) = .5(q_{211} - 10)^2,$$

with the opportunity cost functions being:

$$oc_{111}(\pi_{111}) = \pi_{111}^2, \quad oc_{211}(\pi_{211}) = \pi_{211}^2.$$

Using (24) through (26), we construct the following:

$$F_{111}^1(X) = 2Q_{111} + 1 + \pi_{111} + Q_{111} + .5Q_{211} - .5q_{111} - 100 + Q_{111},$$

$$F_{211}^1(X) = 4Q_{211} + 1 + \pi_{211} + Q_{211} + .5Q_{111} - .5q_{211} - 200 + Q_{211},$$

$$F_{111}^2(X) = q_{111} - 20, \quad F_{211}^2(X) = q_{211} - 10,$$

$$F_{111}^3(X) = -Q_{111} + 2\pi_{111}, \quad F_{211}^3(X) = -Q_{211} + 2\pi_{211}.$$

Solving, as for the first example in Section 2.1, we obtain:

$$Q_{111}^* = 21.00, \quad Q_{211}^* = 30.00,$$

$$q_{111}^* = 20.00, \quad q_{211}^* = 10.00,$$

$$\pi_{111}^* = 10.50, \quad \pi_{211}^* = 15.00.$$

The profits for the service providers are: $U_1^1(X^*) = 875.00$ and $U_2^1(X^*) = 2660.00$, whereas the profit for the network provider, $U_1^2(X^*) = 331.00$.

## 3. The Underlying Dynamics and Stability Analysis

We now describe the underlying dynamics until the equilibrium satisfying variational inequality (22) is achieved. Specifically, we propose a dynamic adjustment process for the evolution of the service providers' service volumes, and that of the network providers' quality levels and transport prices. The users provide feedback through the demand price functions. Observe that, for a current service volume, quality level, and price pattern at time $t$, $X(t) = (Q(t), q(t), \pi(t))$, $-F^1_{ijk}(X(t)) = \frac{\partial U^1_i(Q(t),q(t),\pi(t))}{\partial Q_{ijk}}$, given by minus the expression in (24), is the marginal utility (profit) of service provider $i$ with respect to its service volume to demand market $k$ via network transport provider $j$. Similarly, $-F^2_{ijk}(X(t)) = \frac{\partial U^2_j(Q(t),q(t),\pi(t))}{\partial q_{ijk}}$, given by minus the value in (25), is the network provider $j$'s marginal utility (profit) with respect to its quality level associated with transporting the service volume from $i$ to $k$. Finally, $-F^3_{ijk}(X(t)) = \frac{\partial U^2_j(Q(t),q(t),\pi(t))}{\partial \pi_{ijk}}$, given by minus the value in (26), is the network provider $j$'s marginal utility (profit) with respect to its price charged for transporting the service volume from $i$ to $k$. Below we provide the continuous-time adjustment processes and the corresponding projected dynamical system.

In this framework, the rate of change of the service volume between a service provider $i$ and demand market $k$ via network provider $j$ is in proportion to $-F^1_{ijk}(X)$, as long as the service volume $Q_{ijk}$ is positive. Namely, when $Q_{ijk} > 0$,

$$\dot{Q}_{ijk} = \frac{\partial U^1_i(Q, q, \pi)}{\partial Q_{ijk}}, \tag{27}$$

where $\dot{Q}_{ijk}$ denotes the rate of change of $Q_{ijk}$. However, when $Q_{ijk} = 0$, the nonnegativity condition (3) forces the service volume $Q_{ijk}$ to remain zero when $\frac{\partial U^1_i(Q,q,\pi)}{\partial Q_{ijk}} \leq 0$. Hence, in this case, we are only guaranteed of having possible increases of the service volume. Namely, when $Q_{ijk} = 0$,

$$\dot{Q}_{ijk} = \max\{0, \frac{\partial U^1_i(Q, q, \pi)}{\partial Q_{ijk}}\}. \tag{28}$$

We may write (27) and (28) concisely as:

$$\dot{Q}_{ijk} = \begin{cases} \frac{\partial U^1_i(Q,q,\pi)}{\partial Q_{ijk}}, & \text{if} \quad Q_{ijk} > 0 \\ \max\{0, \frac{\partial U^1_i Q,q,\pi)}{\partial Q_{ijk}}\}, & \text{if} \quad Q_{ijk} = 0. \end{cases} \tag{29}$$

Using similar arguments as above, we may write:

$$\dot{q}_{ijk} = \begin{cases} \frac{\partial U^2_i(Q,q,\pi)}{\partial q_{ijk}}, & \text{if} \quad q_{ijk} > 0 \\ \max\{0, \frac{\partial U^2_i(Q,q,k)}{\partial q_{ijk}}\}, & \text{if} \quad q_{ijk} = 0. \end{cases} \tag{30}$$

and

$$\dot{\pi}_{ijk} = \begin{cases} \frac{\partial U_j^2(Q,q,\pi)}{\partial \pi_{ijk}}, & \text{if} \quad \pi_{ijk} > 0 \\ \max\{0, \frac{\partial U_j^2(Q,q,\pi)}{\partial \pi_{ijk}}\}, & \text{if} \quad \pi_{ijk} = 0. \end{cases} \tag{31}$$

Applying (29), (30), and (31) to all $i = 1, \ldots, m$; $j = 1, \ldots, n$, and $k = 1, \ldots, o$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the service volumes, quality levels, and transport network prices, in vector form, as:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \tag{32}$$

where, since $\mathcal{K}$ is a convex polyhedron, according to Dupuis and Nagurney (1993), $\Pi_{\mathcal{K}}(X, -F(X))$ is the projection, with respect to $\mathcal{K}$, of the vector $-F(X)$ at $X$ defined as

$$\Pi_{\mathcal{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathcal{K}}(X - \delta F(X)) - X}{\delta} \tag{33}$$

with $P_{\mathcal{K}}$ denoting the projection map:

$$P(X) = \text{argmin}_{z \in \mathcal{K}} \|X - z\|, \tag{34}$$

and where $\| \cdot \| = \langle x, x \rangle$. Hence, $F(X) = -\nabla U(Q, q, \pi)$, where $\nabla U(Q, q, \pi)$ is the vector of marginal utilities (profits) with components given by (24), (25), and (26).

We now interpret the ODE (32) in the context of the Cournot-Nash-Bertrand model for a service-oriented Internet with price and quality competition among the network transport providers. First, note that ODE (32) ensures that the service volumes, quality levels, and network prices are always nonnegative. Indeed, if one were to consider, instead, the ordinary differential equation: $\dot{X} = -F(X)$, or, equivalently, $\dot{X} = \nabla U(X)$, such an ODE would not ensure that $X(t) \geq 0$, for all $t \geq 0$, unless additional restrictive assumptions were to be imposed. Moreover, ODE (32) retains the interpretation that if $X$ at time $t$ lies in the interior of $\mathcal{K}$, then the rate at which $X$ changes is greatest when the vector field $-F(X)$ is greatest. In addition, when the vector field $-F(X)$ pushes $X$ to the boundary of the feasible set $\mathcal{K}$, then the projection $\Pi_{\mathcal{K}}$ ensures that $X$ stays within $\mathcal{K}$. Hence, the service volumes, quality levels, and prices are always nonnegative.

Recall now the definition of $F(X)$ (see (24) – (26)) for the integrated model, in which case the *projected dynamical system* (32) states that the rate of change of the service volumes, quality levels, and prices is greatest when the service providers' and network providers' marginal utilities (profits) are greatest. If the marginal utilities with respect to the service volumes are positive, then the service providers will increase their volumes; if they are negative, then they will decrease them. A similar adjustment behavior holds for the network providers in terms of their network transport quality levels and prices. This type of behavior is rational from an economic standpoint. Therefore, ODE (32) is a reasonable continuous adjustment process for the Cournot-Nash-Bertrand game theory model.

Although the use of the projection on the right-hand side of ODE (32) guarantees that the underlying variables are always nonnegative, it also raises the question of existence of a solution to ODE (32), since this ODE is nonstandard due to its discontinuous right-hand side. Dupuis and Nagurney (1993) developed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (32). We cite the following theorem from that paper. See also the book by Nagurney and Zhang (1996).

**Theorem 4**

*$X^*$ solves the variational inequality problem (22) if and only if it is a stationary point of the ODE (32), that is,*

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)). \tag{35}$$

This theorem demonstrates that the necessary and sufficient condition for a pattern $X^* = (Q^*, q^*, \pi^*)$ to be a Cournot-Nash-Bertrand equilibrium, according to Definition 3, is that $X^* = (Q^*, q^*, \pi^*)$ is a stationary point of the adjustment process defined by ODE (32), that is, $X^*$ is the point at which $\dot{X} = 0$.

Consider now the competitive system consisting of the service providers and the network providers, who, in order to maximize their profits, adjust, respectively, their service volumes and their quality level and price patterns by instantly responding to the marginal profits, according to (32). The following questions naturally arise and are of interest. Does the profit (utility) gradient process defined by (32), approach a Cournot-Nash-Bertrand equilibrium, and how does it approach an equilibrium in term of the convergence rate? Also, for a given Cournot-Nash-Bertrand equilibrium, do all the disequilibrium service volumes, quality level, and price patterns that are close to this equilibrium always stay nearby? Motivated by these questions, we now present the stability analysis of Cournot-Nash-Bertrand equilibrium, under the above gradient process.

### 3.1 Stability Under Monotonicity

We now turn to the questions raised above, that is, whether and under what conditions does the adjustment process defined by ODE (32) approaches a Cournot-Nash-Bertrand equilibrium? We first note that Lipschitz continuity of $F(X)$ (cf. Dupuis and Nagurney (1993) and Nagurney and Zhang (1996)) guarantees the existence of a unique solution to

(36) below, where we have that $X^0(t)$ satisfies ODE (32) with service volume, quality level and price pattern $(Q^0, q^0, \pi^0)$. In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} \;=\; \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) \;=\; X^0, \tag{36}$$

with $X^0(0) = X^0$. For convenience, we will sometimes write $X^0 \cdot t$ for $X^0(t)$.

We introduce the following definitions of stability for this adjustment process, which are extensions of those introduced in Zhang and Nagurney (1995) (see also Nagurney and Zhang (1996)). Hereafter, we use $B(X, r)$ to denote the open ball with radius $r$ and center $X$.

We now present some fundamental definitions, for completeness, and some basic qualitative results.

**Definition 4**

*An equilibrium service volume, quality level, and price pattern $X^*$ is stable, if for any $\epsilon > 0$, there exists a $\delta > 0$, such that for all initial $X \in B(X^*, \delta)$ and all $t \geq 0$*

$$X(t) \in B(X^*, \epsilon). \tag{37}$$

*The equilibrium point $X^*$ is unstable, if it is not stable.*

**Definition 5**

*An equilibrium service volume, quality level, and price pattern $X^*$ is asymptotically stable, if it is stable and there exists a $\delta > 0$ such that for all initial service volumes, quality levels, and prices $X \in B(X^*, \delta)$*

$$\lim_{t \to \infty} X(t) \longrightarrow X^*. \tag{38}$$

**Definition 6**

*An equilibrium service volume, quality level, and price pattern $X^*$ is globally exponentially stable, if there exist constants $b > 0$ and $\mu > 0$ such that*

$$\|X^0(t) - X^*\| \;\leq\; b\|X^0 - X^*\|e^{-\mu t}, \quad \forall t \geq 0,\; \forall X^0 \in \mathcal{K}. \tag{39}$$

**Definition 7**

*An equilibrium service volume, quality level, and price pattern* $X^*$ *is a global monotone attractor, if the Euclidean distance* $\|X(t) - X^*\|$ *is nonincreasing in* $t$ *for all* $X \in \mathcal{K}$.

**Definition 8**

*An equilibrium* $X^*$ *is a strictly global monotone attractor, if* $\|X(t) - X^*\|$ *is monotonically decreasing to zero in* $t$ *for all* $X \in \mathcal{K}$.

We now investigate the stability of the adjustment process under various monotonicity conditions.

Recall (cf. Nagurney (1999)) that $F(X)$ is *monotone* if

$$\langle F(X) - F(X^*), X - X^* \rangle \geq 0, \quad \forall X, X^* \in \mathcal{K}. \tag{40}$$

$F(X)$ is *strictly monotone* if

$$\langle F(X) - F(X^*), X - X^* \rangle > 0, \quad \forall X, X^* \in \mathcal{K},\ X \neq X^*. \tag{41}$$

$F(X)$ is *strongly monotone*, if there is an $\eta > 0$, such that

$$\langle F(X) - F(X^*), X - X^* \rangle \geq \eta \|X - X^*\|^2, \quad \forall X, X^* \in \mathcal{K}. \tag{42}$$

The monotonicity of a function $F$ is closely related to the positive-definiteness of its Jacobian $\nabla F$ (cf. Nagurney (1999)). Specifically, if $\nabla F$ is positive-semidefinite, then $F$ is monotone; if $\nabla F$ is positive-definite, then $F$ is strictly monotone; and, if $\nabla F$ is strongly positive-definite, in the sense that the symmetric part of $\nabla F, (\nabla F^T + \nabla F)/2$, has only positive eigenvalues, then $F$ is strongly monotone.

In the context of the Cournot-Nash-Bertrand, where $F(X)$ is the vector of negative marginal utilities as in (24) – (26), we point out that if the utility functions are twice differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions) for the integrated model is positive-definite, then the corresponding $F(X)$ is strictly monotone.

We now present an existence and uniqueness result, the proof of which follows from the basic theory of variational inequalities (cf. Nagurney (1999)).

**Theorem 5**

*Suppose that $F$ is strongly monotone. Then there exists a unique solution to variational inequality (22); equivalently, to variational inequality (23).*

We summarize in the following theorem the stability properties of the utility gradient process, under various monotonicity conditions on the marginal utilities.

**Theorem 6**

**(i).** *If $F(X)$ is monotone, then every Cournot-Nash-Bertrand equilibrium, as defined in Definition 3, provided its existence, is a global monotone attractor for the utility gradient process.*

**(ii).** *If $F(X)$ is strictly monotone, then there exists at most one Cournot-Nash-Bertrand equilibrium. Furthermore, given existence, the unique Cournot-Nash-Bertrand equilibrium is a strictly global monotone attractor for the utility gradient process.*

**(iii).** *If $F(X)$ is strongly monotone, then the unique Cournot-Nash-Bertrand equilibrium, which is guaranteed to exist, is also globally exponentially stable for the utility gradient process.*

**Proof:** The stability assertions follow from Theorems 3.5, 3.6, and 3.7 in Nagurney and Zhang (1996), respectively. The uniqueness in (ii) is a classical variational inequality result, whereas existence and uniqueness as in (iii) follows from Theorem 5. □

### 3.1.1 Examples

We now return to the examples in Sections 2.3.1 and 2.3.2 in order to illustrate some of the above concepts and results.

We begin with the first example in Section 2.3.1. The Jacobian matrix of $F(X) = -\nabla U(Q, q, \pi)$, for this example, denoted by $J(Q_{111}, q_{111}, \pi_{111})$, is

$$J(Q_{111}, q_{111}, \pi_{111}) = \begin{pmatrix} 6 & -1 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

This Jacobian matrix is positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, -$\nabla U(Q, q, \pi)$ is strongly monotone (see also Nagurney (1999)). Thus, both the existence and the uniqueness of the solution to variational inequality (22) with respect to this example are guaranteed. Moreover, the equilibrium solution, which is: $Q^*_{111} = 12$, $q^*_{111} = 1$, and $\pi^*_{111} = 6$ is globally exponentially stable.

The variant of this example, in turn, as described in Section 2.3.1, has the Jacobian matrix of its $F(X) = -\nabla U(Q, q, \pi)$ given by:

$$J(Q_{111}, q_{111}, \pi_{111}) = \begin{pmatrix} 6 & -1 & 1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

We note that this Jacobian matrix is also positive-definite, since it is also strictly diagonally dominant and, hence, the same conclusions as above hold for its equilibrium solution $X^* = (11.82, 0, 5.91)$ and the associated gradient process.

We now turn to the numerical example in Section 2.3.2. The Jacobian matrix of $F(X) = -\nabla U(Q, q, \pi)$, for this example, denoted by $J(Q_{111}, Q_{211}, q_{111}, q_{211}, \pi_{111}, \pi_{211})$, is

$$J(Q_{111}, Q_{211}, q_{111}, q_{211}, \pi_{111}, \pi_{211}) = \begin{pmatrix} 4 & .5 & -.5 & 0 & 1 & 0 \\ .5 & 6 & 0 & -.5 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Given the positive-definiteness of this Jacobian matrix, similar conclusions to those obtained from the two preceding examples follow in terms of existence and uniqueness of the equilibrium $(21, 30, 20, 10, 10.5, 15)$ as well as its stability.

## 4. The Algorithm

As mentioned in Section 3, the projected dynamical system yields continuous-time adjustment processes. However, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories is needed.

We now recall the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993). Specifically, iteration $\tau$ of the Euler method (see also Nagurney and Zhang (1996)) is given by:

$$X^{\tau+1} = P_{\mathcal{K}}(X^\tau - a_\tau F(X^\tau)), \tag{43}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ and $F$ is the function that enters the variational inequality problem (23).

As shown in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996), for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$. Specific conditions for convergence of this scheme as well as various applications to the solutions of other game theory models can be found in Nagurney, Dupuis, and Zhang (1994), Nagurney, Takayama, and Zhang (1995), Cruz (2008), Nagurney (2010), and Nagurney and Li (2013).

### 4.1 Explicit Formulae for the Euler Method Applied to the Cournot-Nash-Bertrand Game Theory Model

The elegance of this procedure for the computation of solutions to our model (in both the dynamic and static, that is, equilibrium, versions) can be seen in the following explicit formulae. In particular, we have the following closed form expression for all the service volumes for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $k = 1, \ldots, o$:

$$Q_{ijk}^{\tau+1} = \max\{0, Q_{ijk}^{\tau} + a_\tau(\hat{\rho}_{ijk}(Q^\tau, q^\tau) + \sum_{h=1}^{n}\sum_{l=1}^{o} \frac{\partial \hat{\rho}_{ihl}(Q^\tau, q^\tau)}{\partial Q_{ijk}} \times Q_{ihl}^{\tau} - \pi_{ijk}^{\tau} - \frac{\partial \hat{f}_i(Q^\tau)}{\partial Q_{ijk}})\}, \quad (44)$$

and the following closed form expression for all the quality levels for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $k = 1, \ldots, o$:

$$q_{ijk}^{\tau+1} = \max\{0, q_{ijk}^{\tau} + a_\tau(-\sum_{h=1}^{m}\sum_{l=1}^{o} \frac{\partial c_{hjl}(Q^\tau, q^\tau)}{\partial q_{ijk}})\} \quad (45)$$

with the explicit formulae for the network transport prices being: for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $k = 1, \ldots, o$:

$$\pi_{ijk}^{\tau+1} = \max\{0, \pi_{ijk}^{\tau} + a_\tau(Q_{ijk}^{\tau} - \frac{\partial oc_{ijk}(\pi_{ijk})}{\partial \pi_{ijk}})\}. \quad (46)$$

We now provide the convergence result. The proof is direct from Theorem 5.8 in Nagurney and Zhang (1996).

**Theorem 7**

*In the Cournot-Nash-Bertrand model for a service-oriented Internet, let $F(X) = -\nabla U(Q, q, \pi)$ be strongly monotone. Also, assume that $F$ is uniformly Lipschitz continuous. Then there exists a unique equilibrium service volume, quality level, and price pattern $(Q^*, q^*, \pi^*) \in \mathcal{K}$ and any sequence generated by the Euler method as given by (43) above, where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$ converges to $(Q^*, q^*, \pi^*)$.*

In the next section, we apply the Euler method to compute solutions to several numerical problems.

## 5. Larger Numerical Examples

In this section, we apply the Euler method, described in the preceding section to compute the Cournot-Nash-Bertrand equilibrium for several service-oriented Internet examples. We set the sequence $\{a_\tau\}$=$1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$. The convergence criterion was that the absolute value of the difference of the iterates at two successive iterations was less than or equal to $10^{-4}$. All the variables (service volumes, quality levels, and network provider prices) were initialized to 0.00.

In the examples (cf. Figure 4) there are 3 service providers, 2 network providers, and 2 demand markets. We implemented the algorithm in FORTRAN and used a LINUX system for the computations. All the examples in this section satisfy the conditions for convergence as given in Theorem 7. The computed equilibria for the numerical examples below, which are guaranteed to exist, are unique, since the respective Jacobians of their $-\nabla U(Q, q, \pi)$ are positive-definite, and, hence, the function $F$ that enters the variational inequality (23) for each of these numerical examples is strongly monotone. Moreover, these $F$s are also uniformly Lipschitz continuous since the utility functions have bounded second order partial derivatives.

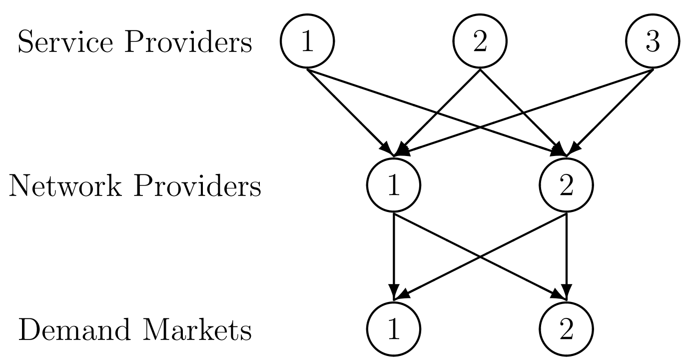


Figure 4: Network Topology for Larger Numerical Examples

### 5.1 Baseline Example 1

The data for the first numerical example in this section, from which we then construct subsequent variants, are as follows.

The production cost functions are:

$$\hat{f}_1(Q) = 2(Q_{111} + Q_{112} + Q_{121} + Q_{122})^2 + (Q_{111} + Q_{112} + Q_{121} + Q_{122}),$$

$$\hat{f}_2(Q) = (Q_{211} + Q_{212} + Q_{221} + Q_{222})^2 + (Q_{211} + Q_{212} + Q_{221} + Q_{222}),$$

$$\hat{f}_3(Q) = 3(Q_{311} + Q_{312} + Q_{321} + Q_{322})^2 + (Q_{311} + Q_{312} + Q_{321} + Q_{322}).$$

The demand price functions are:

$$\hat{\rho}_{111}(Q,q) = -Q_{111} - .5Q_{112} + q_{111} + 100, \quad \hat{\rho}_{112}(Q,q) = -2Q_{112} - 1Q_{111} + q_{112} + 200,$$

$$\hat{\rho}_{121}(Q,q) = -2Q_{121} - .5Q_{111} + .5q_{121} + 100, \quad \hat{\rho}_{122}(Q,q) = -3Q_{122} - Q_{112} + .5q_{122} + 150,$$

$$\hat{\rho}_{211}(Q,q) = -1Q_{211} - .5Q_{212} + .3q_{211} + 100, \quad \hat{\rho}_{212}(Q,q) = -3Q_{212} + .8q_{212} + 200,$$

$$\hat{\rho}_{221}(Q,q) = -2Q_{221} - 1Q_{222} + q_{221} + 140, \quad \hat{\rho}_{222}(Q,q) = -3Q_{222} - Q_{121} + q_{221} + 300,$$

$$\hat{\rho}_{311}(Q,q) = -4Q_{311} + .5q_{311} + 230, \quad \hat{\rho}_{312}(Q,q) = -2Q_{312} - Q_{321} + .3q_{312} + 150,$$

$$\hat{\rho}_{321}(Q,q) = -3Q_{321} - Q_{311} + .2q_{321} + 200, \quad \rho_{322}(Q,q) = -4Q_{322} + .7q_{322} + 300.$$

The transportation cost functions are:

$$c_{111}(Q,q) = q_{111}^2 - .5q_{111},\ c_{112}(Q,q) = .5q_{112}^2 - q_{112},\ c_{121}(Q,q) = .1q_{121}^2 - q_{121},\ c_{122}(Q,q) = q_{122}^2,$$

$$c_{211}(Q,q) = .1q_{211}^2 - q_{211},\ c_{212}(Q,q) = q_{212}^2 - .5q_{212},\ c_{221}(Q,q) = 2q_{221}^2,\ c_{222}(Q,q) = .5q_{222}^2 - q_{222},$$

$$c_{311}(Q,q) = q_{311}^2 - q_{311},\ c_{312}(Q,q) = .5q_{312}^2 - q_{312},\ c_{321}(Q,q) = q_{321}^2 - q_{321},\ c_{322}(Q,q) = 2q_{322}^2 - 2q_{322}.$$

The opportunity cost functions are:

$$oc_{111}(\pi_{111}) = 2\pi_{111}^2,\ oc_{112}(\pi_{112}) = 2\pi_{112}^2,\ oc_{121}(\pi_{121}) = \pi_{121}^2,\ oc_{122}(\pi_{122}) = .5\pi_{122}^2,$$

$$oc_{211}(\pi_{211}) = \pi_{211}^2,\ oc_{212}(\pi_{212}) = .5\pi_{212}^2,\ oc_{221}(\pi_{221}) = 2\pi_{221}^2,\ oc_{222}(\pi_{222}) = 1.5\pi_{222}^2,$$

$$oc_{311}(\pi_{311}) = \pi_{311}^2,\ oc_{312}(\pi_{312}) = 2.5\pi_{312}^2,\ oc_{321}(\pi_{321}) = 1.5\pi_{321}^2,\ oc_{322}(\pi_{322}) = \pi_{322}^2.$$

The Euler method converged in 432 iterations and yielded the approximation to the equilibrium solution reported in Table 2.

The profit of service provider 1 is: 2402.31 and that of service provider 2: 6086.77 and of service provider 3: 3549.49. The profit of network provider 1 is: 184.04 and that of network provider 2: 241.54.

It is interesting to see that demand market 1 obtains no services from service provider 1 since $Q_{111}^*$ and $Q_{121}^*$ are equal to 0.00 and only obtains services from service providers 2 and 3. Demand market 2, however, obtains services from all three service providers. Network provider 1 handles positive volumes of services from all service providers as does network provider 2. It is also interesting to see that two of the quality levels are equal to zero.

Table 2: Equilibrium Solution for the Baseline Example 1

| Service Provider $i$ | Network Provider $j$ | Demand Market $k$ | $Q^*_{ijk}$ | $q^*_{ijk}$ | $\pi^*_{ijk}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.00 | 0.25 | 0.00 |
| 1 | 1 | 2 | 22.67 | 1.00 | 5.67 |
| 1 | 2 | 1 | 0.00 | 5.00 | 0.00 |
| 1 | 2 | 2 | 3.24 | 0.00 | 3.24 |
| 2 | 1 | 1 | 0.00 | 5.00 | 0.00 |
| 2 | 1 | 2 | 14.53 | 0.25 | 14.53 |
| 2 | 2 | 1 | 2.24 | 0.00 | 0.56 |
| 2 | 2 | 2 | 31.97 | 1.00 | 10.66 |
| 3 | 1 | 1 | 7.55 | 0.50 | 3.77 |
| 3 | 1 | 2 | 0.00 | 1.00 | 0.00 |
| 3 | 2 | 1 | 4.18 | 0.50 | 1.39 |
| 3 | 2 | 2 | 15.80 | 0.50 | 7.90 |

Noting that $Q^*_{111} = 0.00$ we then constructed Variant 1 as described below.

### 5.2 Example 2: Variant 1 of Example 1

In this example, we explored the effects of a change in the price function $\rho_{111}$ since recall that, in Example 1, $Q^*_{111} = 0.00$. Such a change in a price function could occur, for example, through enhanced marketing. Specifically, we sought to determine the change in the equilibrium pattern if the consumers at demand market 1 are willing to pay more for the services of the service provider 1 and network provider 1 combination. The new demand price function is:

$$\hat{\rho}_{111}(Q, q) = -Q_{111} - .5Q_{112} + q_{111} + 200,$$

with the remainder of the data as in Example 1. The new computed solution is reported in Table 3. The algorithm converged in 431 iterations.

The profit of service provider 1 is now: 3168.18. The profits of the other two service providers remain as in Example 1. The profit of network provider 1 is now: 209.85 and that of network provider 2: 236.35. Hence, both service provider 1 and network provider 1 have higher profits than in Example 1 and the service volume $Q^*_{111}$ increases from 0.00 to 25.40. There is a reduction in service volume $Q^*_{112}$ and in $Q^*_{122}$.

### 5.3 Example 3: Variant 2 of Example 1

In the next example, we returned to Example 1 and modified all of the transportation cost

Table 3: Equilibrium Solution for Example 2: Variant 1 of Example 1

| Service Provider $i$ | Network Provider $j$ | Demand Market $k$ | $Q^*_{ijk}$ | $q^*_{ijk}$ | $\pi^*_{ijk}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 25.40 | 0.25 | 6.35 |
| 1 | 1 | 2 | 8.67 | 1.00 | 2.17 |
| 1 | 2 | 1 | 0.00 | 4.45 | 0.00 |
| 1 | 2 | 2 | 0.37 | 0.00 | 0.37 |
| 2 | 1 | 1 | 0.00 | 4.45 | 0.00 |
| 2 | 1 | 2 | 14.52 | 0.25 | 14.53 |
| 2 | 2 | 1 | 2.24 | 0.00 | 0.56 |
| 2 | 2 | 2 | 31.97 | 1.00 | 10.66 |
| 3 | 1 | 1 | 7.55 | 0.50 | 3.77 |
| 3 | 1 | 2 | 0.00 | 1.00 | 0.00 |
| 3 | 2 | 1 | 4.18 | 0.50 | 1.39 |
| 3 | 2 | 2 | 15.80 | 0.50 | 7.90 |

functions to include an additional term: $Q_{ijk}q_{ijk}$ to reflect that cost could depend on both congestion level and on quality of transport. The solution obtained via the Euler method for this example is given in Table 4. The Euler method required 705 iterations for convergence.

Table 4: Equilibrium Solution for Example 3: Variant 2 of Example 1

| Service Provider $i$ | Network Provider $j$ | Demand Market $k$ | $Q^*_{ijk}$ | $q^*_{ijk}$ | $\pi^*_{ijk}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.00 | 0.25 | 0.00 |
| 1 | 1 | 2 | 22.52 | 0.00 | 5.63 |
| 1 | 2 | 1 | 0.00 | 4.98 | 0.00 |
| 1 | 2 | 2 | 3.31 | 0.00 | 3.31 |
| 2 | 1 | 1 | 0.00 | 4.99 | 0.00 |
| 2 | 1 | 2 | 14.52 | 0.00 | 14.52 |
| 2 | 2 | 1 | 2.31 | 0.00 | 0.58 |
| 2 | 2 | 2 | 31.84 | 0.00 | 10.61 |
| 3 | 1 | 1 | 7.53 | 0.00 | 3.77 |
| 3 | 1 | 2 | 0.00 | 1.00 | 0.00 |
| 3 | 2 | 1 | 4.19 | 0.00 | 1.40 |
| 3 | 2 | 2 | 15.77 | 0.00 | 7.89 |

The profit of service provider 1 is now: 2380.87. The profit of service provider 2 is: 6053.76 and that of service provider 3 is: 3541.93. The profit of network provider 1 is now: 181.89 and that of network provider 2: 237.21.

Observe that, in this, as in the previous two examples, if $Q^*_{ijk} = 0$, then the price $\pi^*_{ijk} = 0$, which is reasonable. It is interesting to note that, in this example, the inclusion of an additional term $Q_{ijk}q_{ijk}$ to each transportation cost function $c_{ijk}$, with the remainder of the data as in Example 1, results in a decrease in the quality levels in eight out of the twelve computed equilibrium variable values, with the other quality values remaining unchanged. Note that, as mentioned in the introduction, a quality level of zero implies that no extra features are provided. Having an effective modeling and computational framework allows one to explore the effects of changes in the underlying functions on the equilibrium pattern to gain insights that may not be apparent from smaller scale, analytical solutions.

### 5.3 Example 4: Variant 3 of Example 1

In the next example, we used the same data as in Example 3 but we reduced the production cost function for service provider 1 to see the effects on its service volumes (observe that in Table 4, two of its equilibrium service volumes were 0.00).

The new production cost function is:

$$\hat{f}_1(Q) = (Q_{111} + Q_{112} + Q_{121} + Q_{122})^2 + (Q_{111} + Q_{112} + Q_{121} + Q_{122}),$$

The algorithm converged in 638 iterations and yielded the equilibrium pattern reported in Table 5.

Table 5: Equilibrium Solution for Example 4: Variant 3 of Example 1

| Service Provider $i$ | Network Provider $j$ | Demand Market $k$ | $Q^*_{ijk}$ | $q^*_{ijk}$ | $\pi^*_{ijk}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 2.44 | 0.00 | 0.61 |
| 1 | 1 | 2 | 27.48 | 0.00 | 6.87 |
| 1 | 2 | 1 | 4.00 | 0.00 | 2.00 |
| 1 | 2 | 2 | 5.96 | 0.00 | 5.96 |
| 2 | 1 | 1 | 0.00 | 4.98 | 0.00 |
| 2 | 1 | 2 | 14.59 | 0.00 | 14.59 |
| 2 | 2 | 1 | 2.55 | 0.00 | 0.64 |
| 2 | 2 | 2 | 31.28 | 0.00 | 10.43 |
| 3 | 1 | 1 | 7.54 | 0.00 | 3.77 |
| 3 | 1 | 2 | 0.00 | 1.00 | 0.00 |
| 3 | 2 | 1 | 4.19 | 0.00 | 1.40 |
| 3 | 2 | 2 | 15.77 | 0.00 | 7.89 |

The profit of service provider 1 has increased and is now: 3245.84. The profit of service provider 2 has been reduced to 5933.49. The profit of service provider 3 remains unchanged and is 3541.93. The profit of network provider 1 has increased to 214.54 and that of network provider 2 has also increased to 247.33.

Observe that service provider 1, by lowering his production costs, now has all positive equilibrium service volumes.

## 6. Summary and Conclusions

In this paper we developed a game theory model for a service-oriented Internet. The motivation for the research stems, in part, from a need to understand the underlying economics of a service-oriented Internet with more choices as well as to demonstrate the integration of complex competitive behaviors on multitiered networks. We developed both static and dynamic versions of the Cournot-Nash-Bertrand game theory model in which the service providers offer differentiated, but substitutable, services and the network providers transport the services to consumers at the demand markets. Consumers respond to the composition of service and network provision choices and to the quality levels and service volumes, through the prices. The service providers compete in a Cournot-Nash manner, whereas the network providers compete a la Bertrand in prices charged for the transport of the services, as well as with the quality levels associated with the transport.

We derived the governing equilibrium conditions of the integrated game theory model and showed that it satisfies a variational inequality problem. We then described the underlying dynamics, using the theory of projected dynamical systems, and also presented stability analysis and other qualitative results. An algorithm was presented, along with convergence results, which provides a discrete-time version of the continuous-time adjustment processes for the service volumes, quality levels, and prices. We demonstrated the generality of the modeling and computational framework with several numerical examples.

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