A Multitiered Framework for Competitive Flexible Production Value Networks under Environmental Regulations

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Abstract

This paper introduces a novel framework for Flexible Production Value Networks (FPVNs) based on a variational inequality formulation that captures the decentralized behavior of suppliers, manufacturers, distributors, and demand markets under environmental and capacity constraints. The model establishes existence and uniqueness conditions for the equilibrium and is solved through a projection-type algorithm. Numerical illustrations on test networks demonstrate the effectiveness of the approach. In particular, the results show that the equilibrium solutions are stable with respect to demand and cost perturbations, and they highlight the role of environmental constraints in reshaping production and distribution strategies.
Keywords: Flexible Production Value Networks; Supply Chains; Game Theory; Generalized Nash Equilibrium; Variational Inequalities; Stochastic Demands.

1 Introduction

In recent years, the increasing complexity of global supply chains has underscored the need for advanced models that capture the interplay between production flexibility, network structure, and value creation across multiple decision-making levels (see [34], [43]). Within this context, the concept of a Flexible Production Value Network (FPVN) emerges as a unifying framework that integrates the decentralized behavior of suppliers, manufacturers, retailers, and consumers, while addressing both economic competitiveness and environmental sustainability (see [3], [16]).

FPVNs are crucial in the modern industrial landscape, where adaptability and dynamic interactions enable firms to respond effectively to volatile demand, resource constraints, and evolving environmental regulations (see [7], [61]. Unlike traditional linear supply chains, FPVNs emphasize resilience and decentralized decision-making, promoting efficiency, innovation, and sustainability (see [60], [5].

The literature provides a wide range of approaches to modeling flexibility and equilibrium in production and supply chain networks. Early works on flexible manufacturing systems and adaptive scheduling (see [62], [59]) primarily addressed operational flexibility under uncertainty, but did not capture the competitive and networked structure of global supply chains. This gap was progressively addressed by network equilibrium models based on variational inequality theory [34, 43], which introduced decentralized decision-making and equilibrium formulations into supply chain analysis.

More recent contributions have expanded this line of research in different directions. Fargetta and Scrimali [18] extended the framework to dynamic closed-loop systems, explicitly incorporating sustainability objectives. Besik, Nagurney, and Dutta [4] analyzed agricultural supply chains, showing how competition and product quality influence equilibrium outcomes. These studies demonstrate the richness of VI-based approaches, but they tend to address either efficiency, or sustainability, or competition in isolation. None of them, however, considers multitiered supply chain networks where heterogeneous environmental regulations apply simultaneously across different actors.

The FPVN framework proposed in this paper addresses this gap. It extends Flexible Production Networks by explicitly accounting for "value" not only as profit, but also as compliance with environmental regulations and responsiveness to consumer needs (see [3], [16]). In this way, FPVNs provide a comprehensive, value-driven foundation for analyzing sustainable and competitive supply chains.

From an industry standpoint, FPVNs offer a strategic tool for reducing inefficiencies while ensuring regulatory compliance. By incorporating stochastic demand and environmental constraints, they enable firms to avoid losses due to overproduction or underproduction and minimize penalties linked to emission thresholds (see [3], [16]). In this sense, sustainability becomes a source of competitive advantage rather than a regulatory burden (see [28], [56]).

Several global companies already adopt principles consistent with FPVNs. Unilever leverages predictive analytics to optimize production and distribution under environmental regulations [66]. Procter & Gamble operates a decentralized global network embedding sustainability across its tiers [54]. Toyota integrates suppliers and partners across multiple levels with a strong focus on emission reduction [65]. Ikea manages a global supplier network that monitors emissions and resource use [22], while Apple enforces ambitious sustainability requirements across its supply base [2]. These industrial practices align with FPVN principles, confirming their practical relevance.

This paper presents a network-based multitiered game theory model for sustainable competitive FPVNs. The model integrates economic, environmental, and operational constraints to provide a comprehensive approach to decision-making in noncooperative competitive environments. By considering factors such as raw material procurement, production processes, transportation logistics, and market demand fluctuations, the framework aims to maximize profitability while minimizing waste and emissions. In [3] the author highlights the importance of incorporating sustainability into supply chains, advocating for green logistics and production planning. Similarly, in [16] the authors review best practices in green supply chain management, identifying key trends and regulatory influences. Through a game-theoretic approach, the study examines the interplay between different actors in the FPVN, highlighting how individual strategic choices impact overall system performance. In [25] Nagarajan and Sosic explore cooperation and competition in supply chains, demonstrating how game theory can optimize pricing and contract negotiations. Additionally, in [63] and [21] the authors emphasize risk management strategies in supply chains, discussing methods to build resilience against fluctuations and disruptions. The inclusion of stochastic elements in demand and supply further enhances the model's applicability to real-world scenarios. In [61] the authors discuss the role of digital transformation in supply chain design, showcasing how technological advancements can improve logistics and risk mitigation. Furthermore, in [28] Nagurney provides an economic perspective on supply chain networks, detailing pricing strategies and demand-supply interactions. The authors of [56], in turn, construct a supply chain network equilibrium model with multiple decision-makers associated with different tiers and with multiple transportation modes for shipment of the good between tiers using variational inequality theory. The model formulation captures supply-side risk as well as demand-side risk, along with uncertainty in transportation and other costs. The model includes the individual attitudes towards disruption risks of both the manufacturers and the retailers, with the demands for the product associated with the retailers being random.

In this paper, we adopt a variational inequality (VI) approach to model the equilibrium behavior of agents within the Flexible Production Value Network (FPVN). This formalism allows us to represent decentralized decision-making under shared constraints, where agents—such as suppliers, manufacturers, and retailers—optimize individual objectives while interacting through common resources and market feedback. The VI framework is particularly well-suited for capturing such noncooperative equilibria in complex networked systems, as it generalizes classical optimization to include equilibrium and complementarity conditions. It also enables the incorporation of realistic features such as capacity bounds, nonlinearity in costs, and stochastic demand responses, making it an effective tool for both theoretical analysis and computational implementation. The proposed methodology contributes to the ongoing discourse on sustainable industrial practices, offering insights for policymakers, industry leaders, and researchers in the field of operations management.

In the context of the proposed framework, the concept of flexibility is made explicit along several dimensions. First, the model incorporates process flexibility, as firms at different tiers can adjust production levels and resource allocations in response to stochastic demand and environmental regulations. Second, the framework captures product flexibility, since producers can adapt output quantities and product flows in order to meet market requirements under heterogeneous constraints. Moreover, the network structure embeds a form of location flexibility, reflecting the decentralized configuration of suppliers, manufacturers, and retailers and their ability to redistribute activities across tiers. Other dimensions of transformability, such as modularity, universality, and mobility, are not explicitly modeled, although the variational inequality formulation ensures a high degree of scalability, allowing the FPVN framework to be applied to networks of different sizes and complexities. This combination of flexibilities highlights the capacity of the FPVN model to balance efficiency, adaptability, and sustainability in competitive environments.

The novelty of our framework lis in the incorporation of distinct emission regulations across multiple tiers (the suppliers, the manufacturers, and the retailers) of the FPVN. This results in multiple Generalized Nash Equilibrium (GNE) problems, which, to date, have not been investigated in multitiered supply chain network equilibrium problems (cf. [34], [24], [45], [4]). In fact, there are only very few GNE models using variational inequality formulations in supply chain networks and these have been quite recent and in the context of disaster relief (see [36], [31]), healthcare (cf. [35], [47], [30]), and general supply chains [51]. The modeling of such environmental regulations adds to the literature on sustainability and provides for flexibility in terms of the imposition of restrictions. Environmental regulations and policies that have been modeled using variational inequalities have included pollution permits in transportation networks (see [26] and [44]) and in oligopolistic markets (see [32], [33]) as well as carbon taxes (cf. [39]). Sustainable supply chain network models have also been constructed using variational inequality theory in [48], [40], [42], and [1], [50]. Related work has included electronic recycling (see [49] and [67]) and also green investments [55] and [57]. None of these contributions, however, have been GNE problems, and, moreover, multitiered GNE problems, the focus in this paper, and for sustainable competitive FPVNs. See [8], [9], [10], [17], [29], [38] and [37] for a broader perspective on the use of variational inequality models in related domains.

The main contributions of this paper can be summarized as follows:

- introduction of a novel multitiered framework for FPVNs that explicitly incorporates heterogeneous environmental regulations across suppliers, manufacturers, and retailers, representing these interactions within a GNE structure;
- formulation of the problem as a variational inequality, establishing theoretical conditions for the existence and uniqueness of equilibrium solutions, thereby extending classical supply chain models to decentralized and sustainability-oriented contexts;
- integration of stochastic demand and environmental penalties into the model, enabling a realistic representation of uncertainty and adaptive decision-making in competitive environments;
- numerical simulations which demonstrate the robustness and applicability of the framework to multitiered networks;
- finally, the proposed model contributes to the literature on sustainable operations management by offering a unified and value-driven approach that supports both economic competitiveness and environmental compliance.

The rest of the paper is organized as follows. In Section 2, we describe the flexible production value network on which our model is based. In Section 2.1, we describe, in a detailed way, the behavior of the raw material suppliers and we derive the associated variational inequality formulation. In Section 2.2 we detail the manufacturers' behavior and we derive the associated variational formulation while Section 2.3 describes the retailers' role in the FVPN. Section 2.4 is devoted to the description of equilibrium conditions at the demand markets and the associated variational inequality. Section 3 presents the comprehensive variational inequality formulation that integrates the equilibrium conditions derived in the previous sections that captures the strategic interactions among all agents and highlights the structural consequences of their decentralized decisions. In Section 4, we present the numerical results of an illustrative simulation in order to emphasize the key aspects of the multitiered game theory model and to validate its effectiveness. Finally, Section 5 is devoted to our conclusions.

2 The Model

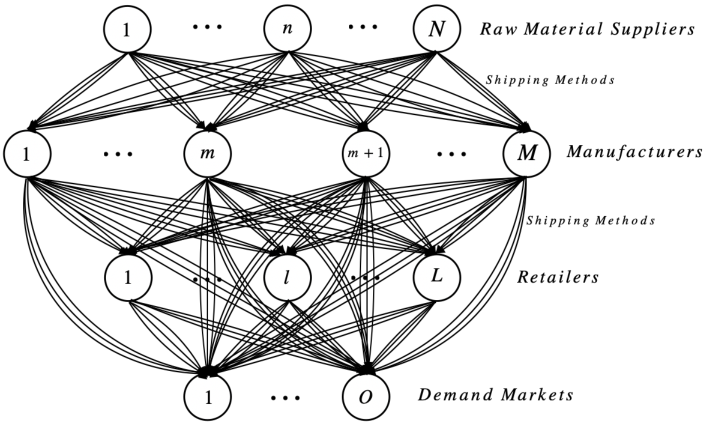


Figure 1: Multitiered Structure of the FPVN

In this section, we present a network-based multitiered game-theoretic model for sustainable and competitive Flexible Production Value Networks (FPVNs). The proposed formulation extends previous supply chain equilibrium models by incorporating stochastic demand through expected deviation functions, as originally introduced in [52] for centralized network design problems. In contrast, the present model considers a decentralized multi-agent environment in which suppliers, manufacturers, and retailers interact strategically within a variational inequality framework. Environmental penalties and flexibility variables are also introduced to capture adaptive and sustainable behavior across the network. Overall, the FPVN integrates economic, environmental, and operational dimensions into a unified decision-making structure for noncooperative competitive environments.

The multitiered architecture of the network is illustrated in Figure 1. The FPVN is composed of four main categories of agents. We consider:

- N raw material suppliers, with the typical one denoted by n;
- M manufacturers, with the typical one denoted by m;
- L retailers, with the typical one denoted by l;
- O demand markets, with the typical one denoted by o.

Raw material suppliers provide essential materials required for production. They determine the quantity of raw materials to produce, the portion to be sold to manufacturers, and any excess that must be stored or discarded. Suppliers incur costs related to production, transportation, and storage, while also considering environmental impact factors such as emissions. Their primary goal is to maximize profit while minimizing waste and emissions. Manufacturers acquire raw materials and process them into finished products. They must balance procurement decisions, production levels, and sales strategies while managing storage costs and transportation logistics. They face additional complexity due to uncertain demand from both retailers and direct customers, leading to potential overproduction or underproduction. Consequently, penalties are introduced to discourage inefficiencies and to promote optimized production planning. Retailers, in turn, purchase finished goods from manufacturers and distribute them to end customers. They decide on inventory levels, purchase volumes, and selling prices while considering stochastic customer demand. Their decision-making process is influenced by transportation costs, storage constraints, and penalties associated with excess inventory or stockouts. Environmental considerations, such as emissions from distribution operations, are also incorporated into their optimization problems. Customers are the final consumers in the FPVN, determining demand for products. Customer demand is treated as a stochastic variable, impacting the strategies of manufacturers and of retailers. Pricing strategies and fulfillment policies are shaped by these demand uncertainties, influencing overall network performance.

Each agent in the FPVN operates within a noncooperative competitive framework, where individual optimization problems are interconnected through pricing mechanisms, demand interactions, and shared environmental constraints. The model integrates key objectives, including profit maximization, environmental sustainability, operational efficiency, and network-wide equilibrium. Each agent seeks to maximize expected profit by balancing revenue generation with production, procurement, transportation, and storage costs. Emission reduction is a fundamental concern across all levels of the FPVN, and agents face regulatory constraints on emissions, requiring them to incorporate sustainability into decision-making. To prevent inefficiencies, penalties are applied for overproduction, underproduction, excess inventory, and unmet demand. These constraints incentivize agents to align production and distribution strategies with actual market conditions. The model establishes a competitive equilibrium by ensuring that individual optimization decisions lead to a stable, sustainable, and economically viable FPVN.

The decision-making process within the FPVN is formalized using Generalized Nash Equilibrium (GNE) and Variational Equilibrium (VE) concepts. Each agent's optimization problem considers network-wide constraints, leading to an equilibrium where no agent can improve its outcome unilaterally. The variational formulation ensures that the collective decisions of all agents satisfy shared resource and environmental limitations. By integrating economic, environmental, and operational factors, this model provides a robust framework for evaluating FPVNs under uncertainty. The insights derived from this approach contribute to the ongoing discourse on sustainable supply chain management, offering practical strategies for policymakers, industry leaders, and researchers.

In our formulation, we distinguish between regular production and additional production at each tier of the FPVN. Regular production represents the baseline capacity of firms, obtained through standard processes and technologies at relatively lower marginal costs and within predefined environmental thresholds. In contrast, additional production reflects the possibility of expanding output beyond the baseline by mobilizing extra resources or adopting more intensive production strategies. This flexibility, however, comes at the price of higher marginal costs and increased environmental impacts, which are explicitly captured in the cost functions and regulatory constraints of the model. In this way, the FPVN framework accounts for the trade-off between efficiency and adaptability: firms can decide whether to remain within their baseline capacity or to exploit additional capacity in order to meet demand surges, at the expense of higher costs and penalties. Non-flexible production networks would not incorporate this distinction, as their capacity would be fixed and inelastic, thus preventing the system from adapting to stochastic demand and heterogeneous regulations.

In the following sections, we describe the behavior of all agents in the network, including raw material suppliers, manufacturers, retailers, and customers, analyzing their decision-making processes and interactions within the framework.

2.1 Raw Material Suppliers' Behavior

Raw material suppliers are responsible for the extraction, processing, and distribution of raw materials to manufacturers. They must decide the quantity of raw material to provide while managing constraints related to production capacity, transportation logistics, and storage limitations. The supply of raw materials is influenced by both deterministic and stochastic factors, including fluctuations in demand from manufacturers, variations in production costs, and external disruptions such as geopolitical events or environmental regulations.

In the FPVN framework, inventory variables represent short-term buffers that allow agents to manage demand uncertainty. These quantities are not intended as long-term storage but as temporary adjustments that maintain network balance when realized demand differs from expected values. This interpretation ensures consistency with the flexible and adaptive nature of FPVNs.

Let $x_{nr} \in \mathbb{R}_+$ be the quantity of raw material of type r, $r = 1, \dots, R$, produced by supplier n, $n = 1, \dots, N$. We denote by $x_n = (x_{nr})_{r=1,\dots,R} \in \mathbb{R}^R_+$ the vector of all types of raw material produced by supplier n, $n = 1, \dots, N$. Associated with the production of x_n, supplier n faces production costs

$$c^1_{nr} \equiv c^1_{nr}(x_{nr}),\ n = 1, \dots, N,\ r = 1, \dots, R.$$

In this competitive framework, each supplier seeks to maximize its expected profit by strategically allocating resources to optimize production and minimize waste. A key challenge faced by suppliers is balancing the trade-off between producing sufficient materials to meet demand while avoiding excessive stock accumulation that could lead to increased storage costs or potential financial losses. To mitigate these risks, suppliers may choose to expand production capacity selectively, incorporating flexibility into their operations to accommodate shifts in market conditions.

Let $y_{nr} \in \mathbb{R}_+$ be the additional raw material of type r, $r = 1, \dots, R$, that can be produced with expansion by raw supplier n, $n = 1, \dots, N$. We denote by $y_n = (y_{nr})_{r=1,\dots,R} \in \mathbb{R}^R_+$ the vector of all types of additional raw material produced by supplier n, $n = 1, \dots, N$. Associated with the increase of capacity production of all types of raw materials, supplier n faces investment costs (purchase of new machinery, extension of existing facilities,...) denoted by

$$c_{nr} \equiv c_{nr}(y_{nr}),\ n = 1, \dots, N,\ r = 1, \dots, R,$$

as well as production costs

$$c^2_{nr} \equiv c^2_{nr}(y_{nr}), n = 1, \dots, N,\ r = 1, \dots, R.$$

We note that the production cost of additional raw material of type r, y_{nr}, is generally different from the cost of producing x_{nr}. Indeed, the cost of the latter is generally lower because production is operating at an optimal rate (see [53]). Additional production may have increasing marginal costs due to bottlenecks in the production chain, increased energy consumption, or the need to operate beyond normal working hours. Furthermore, if the additional material is produced in another facility or secondary plant, it may be subject to higher management, logistics, and other costs.

Finally, let $z_{nr} \in \mathbb{R}_+$ be the quantity of raw material of type r left unsold, $n = 1, \dots, N$ and $r = 1, \dots, R$, and we denote the cost for holding unsold raw material of type r faced by supplier n by

$$c^d_{nr} \equiv c^d_{nr}(z_{nr}),\ n = 1, \dots, N,\ r = 1, \dots, R.$$

As before, for each supplier n, $n = 1, \dots, N$, we denote by $z_n = (z_{nr})_{r=1,\dots,R} \in \mathbb{R}^R_+$ the vector of all types of raw materials unsold. We note that unsold quantities z_{nr} implicitly account for short-term warehousing or temporary storage at the suppliers' facilities. This choice keeps the model analytically tractable while representing the possibility of holding limited inventories before further distribution.

Transportation plays a crucial role in the raw material supply process, as suppliers must determine the most cost-effective and sustainable means of delivering materials to manufacturers. Different transportation modes introduce varying costs and emission levels, affecting both financial and environmental performance. Suppliers must evaluate these trade-offs when selecting distribution strategies, ensuring that transportation expenses and environmental impact remain within acceptable limits.

Let $x_{nrmi} \in \mathbb{R}_+$ be the quantity of raw material of type r transported from supplier n to manufacturer m using transportation mode i and let $y_{nrmi} \in \mathbb{R}_+$ be the quantity of additional raw material of type r due to expansion transported from supplier n to manufacturer m using transportation mode i, $n = 1, \dots, N,\ r = 1, \dots, R, m = 1, \dots, M,\ i = 1 \dots, I$. We denote the transportation costs for regular and additional raw material of type r, respectively, by

$$c^{1t}_{nrmi} \equiv c^{1t}_{nrmi}(x_{nrmi}),\ \forall n = 1, \dots, N,\ r = 1, \dots, R,\ m = 1, \dots, M,\ i = 1, \dots, I,$$

and

$$c^{2t}_{nrmi} \equiv c^{2t}_{nrmi}(y_{nrmi}),\ \forall n = 1, \ldots, N,\ r = 1, \ldots, R,\ m = 1, \ldots, M,\ i = 1, \ldots, I.$$

We note that the transportation costs for regular and additional raw materials may differ, even when the same transport vehicle is used, due to several factors. Although the vehicle remains the same, the handling, loading, and unloading processes for the additional materials may be more complex, requiring more time and potentially specialized equipment. Additionally, the storage requirements for the extra materials might differ, resulting in higher labor costs. Furthermore, the added weight or different routing considerations could lead to increased fuel consumption, affecting overall fuel efficiency.

Environmental sustainability is a critical aspect of raw material supply chains. Suppliers must account for the emissions generated during production, transportation, and storage. Regulatory policies impose strict emission limits, requiring suppliers to adopt cleaner production methods and to optimize transportation strategies to reduce their environmental footprint. Additionally, financial penalties for exceeding emission thresholds provide further incentives for suppliers to integrate sustainability into their decision-making processes. In this context, we denote the environmental emissions associated with the production of regular and additional raw materials of type r, the emissions associated with transportation of regular and additional raw materials of type r to a generic production facility m using transport mode i, and those associated with the storage of unsold raw material of type r, respectively, by:

$$e^1_{nr} \equiv e^1_n(x_{nr}), \forall n = 1, \ldots, N,\ r = 1, \ldots, R,$$

$$e^2_{nr} \equiv e^2_n(y_{nr}), \forall n = 1, \ldots, N,\ r = 1, \ldots, R,$$

$$e^1_{nrmi} \equiv e^1_{nrmi}(x_{nrmi}), \forall n = 1, \ldots, N,\ r = 1, \ldots, R,\ m = 1, \ldots, M,\ i = 1, \ldots, I,$$

$$e^2_{nrmi} \equiv e^2_{nrmi}(y_{nrmi}), \forall n = 1, \ldots, N,\ r = 1, \ldots, R,\ m = 1, \ldots, M,\ i = 1, \ldots, I,$$

and

$$e^d_{nr} \equiv e^d_{nr}(z_{nr}), \forall n = 1, \ldots, N,\ r = 1, \ldots, R.$$

Although the current formulation does not include a separate term for warehousing-related emissions, such effects can be interpreted as part of the functions $e^d_{nr}(z_{nr})$, which already capture energy use and environmental impact associated with temporary storage and handling. This assumption maintains model simplicity while preserving its environmental consistency.

We assume that the above cost and emission functions are convex and continuously differentiable with respect to their own variables. Decision variables, cost and emissions functions for raw material supplier n, $n = 1, \ldots, N$, are listed in Table A.1 and Table A.2 in Appendix A.

Given the inherent uncertainties in demand and supply chain operations, suppliers must adopt robust risk management strategies. Stochastic demand from the lower level of the network introduces variability in procurement volumes, which suppliers must address by implementing flexible production and inventory policies. Risk mitigation approaches, such as diversifying sourcing strategies or leveraging predictive analytics for demand forecasting, help suppliers maintain stability and competitiveness in volatile market conditions.

Raw material suppliers are modeled as rational and intelligent agents engaging in a noncooperative competition within a Generalized Nash Equilibrium (GNE) framework. Each supplier independently optimizes its decision variables—production levels, transportation strategies, and inventory management—to maximize its expected profit while minimizing costs associated with production, transportation, and storage. However, their strategic interactions are constrained by a shared environmental regulation: total emissions across all suppliers must not exceed a common regulatory threshold. This introduces a coupling constraint, making the equilibrium problem more complex, since each supplier's optimal strategy depends not only on market conditions but also on the emission levels resulting from the decisions of all other suppliers in the network (see [13],[30], [31],[46]).

Each raw material supplier n, $n = 1, \ldots, N$, thus faces the following constrained optimization problem, where it aims to maximize its expected utility, represented by a weighted objective function that balances total expected profit—derived from raw material sales to manufacturers minus production, investment, and transportation costs—and total environmental emissions, as previously discussed:

$$\max E(U_n) = \max \Bigg\{ \sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I} p^{1*}_{nrmi} x_{nrmi} + \sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I} \rho^{1*}_{nrmi} y_{nrmi} - \sum_{r=1}^{R} c^1_{nr}(x_{nr}) - \sum_{r=1}^{R} c^2_{nr}(y_{nr})$$
$$- \sum_{r=1}^{R} c_{nr}(y_{nr}) - \sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I} c^{1t}_{nmi}(x_{nmi}) - \sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I} c^{2t}_{nmi}(y_{nmi}) - \sum_{r=1}^{R} c^d_{nr}(z_{nr})$$
$$-\alpha_n \sum_{r=1}^{R} \left[e^1_{nr}(x_{nr}) + e^2(y_{nr}) + e^d_{nr}(z_{nr}) + \sum_{m=1}^{M}\sum_{i=1}^{N} e^1_{nrmi}(x_{nrmi}) + \sum_{m=1}^{M}\sum_{i=1}^{N} e^2_{nrmi}(y_{nrmi}) \right] \Bigg\} \tag{1}$$

subject to constraints:

$$x_{nr} + y_{nr} + \underline{z}_{nr} = \sum_{m=1}^{M}\sum_{i=1}^{I} x_{nrmi} + \sum_{m=1}^{M}\sum_{i=1}^{I} y_{nrmi} + z_{nr}, \ \forall r = 1, \ldots, R \tag{2}$$

$$\sum_{r=1}^{R} x_{nr} \le \mathcal{Q}^1_n \tag{3}$$

$$\sum_{r=1}^{R} y_{nr} \le \mathcal{Q}^2_n \tag{4}$$

$$x_{nr}, y_{nr}, z_{nr}, x_{nrmi}, y_{nrmi} \ge 0, \quad \forall r = 1, \ldots, R, \ m = 1, \ldots, M, \ i = 1, \ldots, I, \tag{5}$$

$$\sum_{n=1}^{N}\sum_{r=1}^{R} \left[e^1_{nr}(x_{nr}) + e^2_{nr}(y_{nr}) + \sum_{m=1}^{M}\sum_{i=1}^{I} e^1_{nrmi}(x_{nrmi}) + \sum_{m=1}^{M}\sum_{i=1}^{I} e^2_{nrmi}(y_{nrmi}) + e^d_n(z_{nr}) \right] \le \varepsilon. \tag{6}$$

Objective function (1) represents the expected utility of Raw Material Supplier n, $n = 1, \ldots, N$, given by the weighted sum of the total expected profit and the total environmental emissions, where p^{1*}_{nrmi} and ρ^{1*}_{nrmi} represent the price obtained from the sale of a unit of raw material of type r, $r = 1, \ldots, R$, to a manufacturer m, $m = 1, ..., M$, with the shipment method i, $i = 1, ..., I$, and the price obtained from the sale of a unit of additional raw material of type r, $r = 1, \ldots, R$, to manufacturer m, respectively. In this formulation, $\alpha_n > 0$ denotes a parameter that allows us to express environmental emissions in terms of costs. Moreover, the parameter α_n enables distinct raw material suppliers to have different emissions in terms of costs.

Constraint (2) ensures that the total quantity of raw material produced, including the additional production capacity, is either sold to manufacturers, stored as unsold inventory or discarded, i.e. $\underline{z}_n$. It maintains consistency in the material flow within the supply chain and represents a material balance constraint.

Constraints (3) and (4) impose upper bounds on the quantity of raw material that a supplier can produce under normal operating conditions ($\mathcal{Q}^1_n$) and the maximum additional capacity ($\mathcal{Q}^2_n$) that can be generated through expansion. These limits reflect the supplier's physical and technological production capabilities.

Constraint (5) represents the domain of decision variables for a Raw Material Supplier n.

Constraint (6) accounts for the total environmental impact of raw material suppliers by aggregating emissions from production, transportation, and storage. Since emissions contribute to a shared regulatory limit ε, this constraint introduces interdependence among suppliers, making it a coupling constraint in the Generalized Nash Equilibrium framework.

We observe that, conservation law (2) allows us to express vector variable z_{nr}, $n = 1, \ldots, N$, $r = 1, \ldots, R$, in terms of the vector variable $(x_{nr}, y_{nr}, x^1_{nr}, y^1_{nr})$. Therefore, we can rewrite the constrained optimization problem (1)-(6) of each raw material supplier n as follows:

$$\max E(U_n) = \max \Bigg\{ \sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I} p^{1*}_{nmi} x_{nmi} + \sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I} \rho^{1*}_{nmi} y_{nmi} - \sum_{r=1}^{R} c^1_{nr}(x_{nr}) - \sum_{r=1}^{R} c^2_n(y_{nr})$$
$$- \sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I} c^{1t}_{nrmi}(x_{nrmi}) - \sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I} c^{2t}_{nrmi}(y_{nrmi}) - \sum_{r=1}^{R} c^d_{nr}(x_{nr}, y_{nr}, x^1_{nr}, y^1_{nr})$$
$$-\alpha_n \sum_{r=1}^{R} \Bigg[e^1_{nr}(x_{nr}) + e^2_{nr}(y_{nr}) + e^d_{nr}(x_{nr}, y_{nr}, x^1_{nr}, y^1_{nr}) + \sum_{m=1}^{M}\sum_{i=1}^{N} e^1_{nrmi}(x_{nrmi}) + \sum_{m=1}^{M}\sum_{i=1}^{N} e^2_{nrmi}(y_{nrmi}) \Bigg] \Bigg\} \tag{7}$$

subject to constraints:

$$\sum_{r=1}^{R} x_{nr} \le \mathcal{Q}^1_n \tag{8}$$

$$\sum_{r=1}^{R} y_{nr} \le \mathcal{Q}^2_n \tag{9}$$

$$x_{nr}, y_{nr}, x_{nrmi}, y_{nrmi} \ge 0, \quad \forall r = 1, \dots, R, \; m = 1, \dots, M, \; i = 1, \dots, I \tag{10}$$

$$\sum_{n=1}^{N}\sum_{r=1}^{R} \Bigg[e^1_{nr}(x_{nr}) + e^2_{nr}(y_{nr}) + \sum_{m=1}^{M}\sum_{i=1}^{I} e^1_{nrmi}(x_{nrmi}) + \sum_{m=1}^{M}\sum_{i=1}^{I} e^2_{nrmi}(y_{nrmi}) + e^d_{nr}(x_{nr}, y_{nr}, x^1_{nr}, y^1_{nr}) \Bigg] \le \varepsilon. \tag{11}$$

We define the feasible set for each raw material supplier n, $n = 1, \dots, N$ as:

$$K^n = \{\psi_n \equiv (x_n, y_n, x^1_n, y^1_n) \in \mathbb{R}^{2R(1+MI)}_+ | (8) - (10) \text{ hold}\},$$

and we denote by $\mathcal{S}$ the set of shared constraints, namely:

$$\mathcal{S} = \{\psi \equiv (x, y, x^1, y^1) \in \mathbb{R}^{2NR(1+MI)}_+ | (11) \text{ hold}\}.$$

Finally, we define the feasible set $K = \prod_{n=1}^{N} K^n$ and $\mathbb{K} = K \cap \mathcal{S}$.

We refer to objective function (7) as the expected utility $E(U_n)$, $n = 1, \dots, N$. Assuming that (as stated before) the cost and emission functions for each raw material supplier are convex and continuously differentiable, we can state:

Definition 2.1 (Generalized Nash Equilibrium for Raw Materials Suppliers) *A strategy profile $\psi^* \in \mathbb{K}$ is a Generalized Nash Equilibrium if, for any Raw Material Supplier n, $n = 1, \dots, N$*

$$E(U_n(x^*_n, y^*_n, x^{1*}_n, y^{1*}_n, x^*_{-n}, y^*_{-n}, x^{1*}_{-n}, y^{1*}_{-n})) \ge E(U_n(x_n, y_n, x^1_n, y^1_n, x^*_{-n}, y^*_{-n}, x^{1*}_{-n}, y^{1*}_{-n}))$$
$$\forall \psi_n \in K^n, \; \forall \psi \in \mathcal{S}, \tag{12}$$

where

$$x^*_{-n} = (x^*_1, \dots, x^*_{n-1}, x^*_{n+1}, \dots, x^*_N), \; y^*_{-n} = (y^*_1, \dots, y^*_{n-1}, y^*_{n+1}, \dots, y^*_N)$$
$$x^{1*}_{-n} = (x^{1*}_1, \dots, x^*_{n-1}, x^{1*}_{n+1}, \dots, x^{1*}_N), \; y^{1*}_{-n} = (y^{1*}_1, \dots, y^{1*}_{n-1}, y^{1*}_{n+1}, \dots, y^{1*}_N).$$

Each raw material supplier wishes to maximize its expected utility. The above definition states that no raw material supplier, given the circumstances and the strategies of the other suppliers, at equilibrium, is willing unilaterally to change its vector of strategies, because it may end up with a lower expected utility. We observe that the expected utility of each supplier depends not only on its own decisions, but also on the strategies of other suppliers. Also, their feasible sets are interconnected because of the shared constraints on environmental emissions. The latter condition makes the problem a Generalized Nash Equilibrium model, according to [14].

In general, Generalized Nash Equilibrium problems can be formulated through quasi-variational inequality problems (see [19]). However, a class of Generalized Nash Equilibria, the Variational Equilibria, can be formulated as a variational inequality problem, as discussed, for instance, in [13], [30], [31] and [46]. Following the approach presented in these contexts, we will deal with the variational equilibrium of the model.

Definition 2.2 (Variational Equilibrium) *A strategy vector $\psi^* \in \mathbb{K}$ is a Variational Equilibrium of the above Generalized Nash Equilibrium problem if $\psi^* \in \mathbb{K}$ is a solution to the variational inequality:*

$$
\begin{aligned}
&\sum_{n=1}^{N}\sum_{r=1}^{R}\Bigg[\frac{\partial c^1_{nr}(x^*_{nr})}{\partial x_{nr}}+\frac{\partial c^d_{nr}(x^*_{nr},y^*_{nr},x^{1*}_{nr},y^{1*}_{nr})}{\partial x_{nr}}\\
&\qquad+\alpha_n\left(\frac{\partial e^1_{nr}(x^*_{n})}{\partial x_{nr}}+\frac{\partial e^d_{nr}(x^*_{nr},y^*_{nr},x^{1*}_{nr},y^{1*}_{nr})}{\partial x_{nr}}\right)\Bigg]\times(x_{nr}-x^*_{nr})\\
&+\sum_{n=1}^{N}\sum_{r=1}^{R}\Bigg[\frac{\partial c^2_{nr}(y^*_{nr})}{\partial y_{nr}}+\frac{\partial c_{nr}(y^*_{nr})}{\partial y_{nr}}+\frac{\partial c^d_{nr}(x^*_{nr},y^*_{nr},x^{1*}_{nr},y^{1*}_{nr})}{\partial y_{nr}}\\
&\qquad+\alpha_n\left(\frac{\partial e^2_{nr}(y^*_{nr})}{\partial y_{nr}}+\frac{\partial e^d_{nr}(x^*_{nr},y^*_{nr},x^{1*}_{nr},y^{1*}_{nr})}{\partial y_{nr}}\right)\Bigg]\times(y_{nr}-y^*_{nr})\\
&+\sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I}\Bigg[\frac{\partial c^{1t}_{nrmi}(x^*_{nrmi})}{\partial x_{nrmi}}+\frac{\partial c^d_{nr}(x^*_{nr},y^*_{nr},x^{1*}_{nr},y^{1*}_{nr})}{\partial x_{nrmi}}\\
&\qquad+\alpha_n\left(\frac{\partial e^1_{nrmi}(x^*_{nrmi})}{\partial x_{nrmi}}+\frac{\partial e^d_{nr}(x^*_{nr},y^*_{nr},x^{1*}_{nr},y^{1*}_{nr})}{\partial x_{nrmi}}\right)-p^{1*}_{nrmi}\Bigg]\times(x_{nrmi}-x^*_{nrmi})\\
&+\sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I}\Bigg[\frac{\partial c^{2t}_{nrmi}(y^*_{nrmi})}{\partial y_{nrmi}}+\frac{\partial c^d_{nr}(x^*_{nr},y^*_{nr},x^{1*}_{nr},y^{1*}_{nr})}{\partial y_{nrmi}}\\
&\qquad+\alpha_n\left(\frac{\partial e^2_{nrmi}(y^*_{nrmi})}{\partial y_{nrmi}}+\frac{\partial e^d_{nr}(x^*_{nr},y^*_{nr},x^{1*}_{nr},y^{1*}_{nr})}{\partial y_{nrmi}}\right)-\rho^{1*}_{nrmi}\Bigg]\times(y_{nrmi}-y^*_{nrmi})\geq 0,\\
&\qquad\qquad\forall\psi\in\mathbb{K}.
\end{aligned}
\tag{13}
$$

The advantage of constructing a variational equilibrium consists in using the well-known variational inequality theory, for which theorems of existence and uniqueness of the solution are stated (see [27] and [28]).

For simplicity of notation, we now put variational inequality (13) into standard form, that is: determine $X^* \in \mathcal{K}$ such that:

$$\langle F(X^*), X - X^*\rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{14}$$

where $\mathcal{K}$ is a closed and convex set.

To this purpose, we define $X \equiv \psi$, $F(X) = (F^1(X), F^2(X), F^3(X), F^4(X))$ and $\mathcal{K} \equiv \mathbb{K}$, where:

$$
\begin{aligned}
F^1_{nr}(X)&=\Bigg[\frac{\partial c^1_{nr}(x_{nr})}{\partial x_{nr}}+\frac{\partial c^d_{nr}(x_{nr},y_{nr},x^1_{nr},y^1_{nr})}{\partial x_{nr}}\\
&\qquad+\alpha_n\left(\frac{\partial e^1_{nr}(x_{nr})}{\partial x_{nr}}+\frac{\partial e^d_{nr}(x_{nr},y_{nr},x^1_{nr},y^1_{nr})}{\partial x_{nr}}\right)\Bigg],\quad\forall n,\ r;\\
F^2_{nr}(X)&=\Bigg[\frac{\partial c^2_{nr}(y_{nr})}{\partial y_{nr}}+\frac{\partial c_{nr}(y_{nr})}{\partial y_{nr}}+\frac{\partial c^d_{nr}(x_{nr},y_{nr},x^1_{nr},y^1_{nr})}{\partial y_{nr}}\\
&\qquad+\alpha_n\left(\frac{\partial e^2_{nr}(y_{nr})}{\partial y_{nr}}+\frac{\partial e^d_{nr}(x_{nr},y_{nr},x^1_{nr},y^1_{nr})}{\partial y_{nr}}\right)\Bigg],\quad\forall n,\ r;\\
F^3_{nrmi}(X)&=\Bigg[\frac{\partial c^{1t}_{nrmi}(x_{nrmi})}{\partial x_{nrmi}}+\frac{\partial c^d_{nr}(x_{nr},y_{nr},x^1_{nr},y^1_{nr})}{\partial x_{nrmi}}\\
&\qquad+\alpha_n\left(\frac{\partial e^1_{nrmi}(x_{nrmi})}{\partial x_{nrmi}}+\frac{\partial e^d_{nr}(x_{nr},y_{nr},x^1_{nr},y^1_{nr})}{\partial x_{nmi}}\right)-p^1_{nrmi}\Bigg],\quad\forall n,r,m,i;\\
F^4_{nrmi}(X)&=\Bigg[\frac{\partial c^{2t}_{nrmi}(y_{nrmi})}{\partial y_{nrmi}}+\frac{\partial c^d_{nr}(x_{nr},y_{nr},x^1_{nr},y^1_{nr})}{\partial y_{nrmi}}\\
&\qquad+\alpha_n\left(\frac{\partial e^2_{nrmi}(y^*_{nrmi})}{\partial y_{nrmi}}+\frac{\partial e^d_{nr}(x_{nr},y_{nr},x^1_{nr},y^1_{nr})}{\partial y_{nrmi}}\right)-\rho^1_{nrmi}\Bigg],\quad\forall n,r,m,i.
\end{aligned}
\tag{15}
$$

Under the imposed assumptions on the cost and emission functions, we can deduce that the function $F(X)$ defining variational inequality (14) is continuous. Moreover, it is easy to show that the feasible set $\mathcal{K}$ is compact (since constraints (8)-(9) and conservation law (2) hold). Therefore, the existence of a solution to variational inequality (14) is guaranteed by the classical theory of variational inequalities (see [23]).

2.2 Manufacturers' Behavior

Manufacturers play a central role in Flexible Production Value Networks (FPVNs) by transforming raw materials into finished goods that are then distributed to retailers or directly to end customers. Unlike traditional linear supply chains, where production follows a rigid structure, manufacturers in FPVNs operate in a highly dynamic and competitive environment. They must continuously adjust their procurement strategies, production levels, and sales approaches to respond to fluctuating market demand, supply chain disruptions, and sustainability constraints.

In this competitive framework, manufacturers independently decide on raw material procurement, determining the quantity to purchase from suppliers and selecting the most cost-effective and sustainable transportation modes. Once raw materials are acquired, manufacturers must optimize their production processes, ensuring that the output aligns with market demand while minimizing costs and environmental impact. They also face inventory management challenges, as excessive stockpiling leads to increased storage costs, while insufficient production results in lost sales and customer dissatisfaction.

An additional layer of complexity arises from demand uncertainty. Manufacturers must anticipate demand from both retailers, who act as intermediaries in the distribution process, and direct customers, who may purchase finished products through online platforms or other channels. Given that demand can fluctuate due to external factors such as economic conditions, seasonal trends, or market shocks, manufacturers must develop robust production planning strategies that incorporate risk mitigation techniques, such as flexible production capacity, predictive analytics, and adaptive pricing models.

Manufacturers behave as noncooperative competitors with respect to other manufacturers within the same tier, while interacting with suppliers and retailers through equilibrium price and flow relations. Each manufacturer aims to maximize its own profitability, making decisions that optimize production efficiency while considering the actions of competing manufacturers in the network. This decentralized decision-making structure can lead to market inefficiencies, such as overproduction (when supply exceeds actual demand) or underproduction (when production falls short of market needs). Consequently, penalty mechanisms are introduced to discourage inefficiencies and incentivize optimal production planning, ensuring a better alignment between supply and demand while maintaining economic and environmental sustainability.

Each manufacturer m, $m = 1, \dots, M$, must determine the quantity of raw material purchased from supplier n, $n = 1, \dots, N$, using transport mode i, $i = 1, \dots, I$, previously denoted as $x_{nmi} \in \mathbb{R}_+$ and the additional raw material purchased as $y_{nmi} \in \mathbb{R}_+$. The finished product quantity produced by manufacturer m is represented as $g_m \in \mathbb{R}_+$. This product can either be sold to retailer l, $l = 1, \dots, L$ via transport mode $i = 1, \dots, I$, denoted as $g_{mli} \in \mathbb{R}_+$, or sold directly to demand market o, $o = 1, \dots, O$, via transport mode i, denoted as $g_{moi} \in \mathbb{R}_+$. In this context

$$g_m = \sum_{l=1}^{L} \sum_{i=1}^{I} g_{mli} + \sum_{o=1}^{O} \sum_{i=1}^{I} g_{moi}, \quad \forall m = 1, \dots, M. \tag{16}$$

The above conservation law ensures that the total finished product quantity for each manufacturer m, $m = 1, \dots, M$, is equal to the quantities of finished product transacted with all retailers and all demand markets of the network. Therefore, g_m can be expressed in terms of the vectors $g_m^1 \equiv (g_{mli})_{l=1,\dots,L,\ i=1,\dots,I} \in \mathbb{R}_+^{LI}$ and $g_m^2 \equiv (g_{moi})_{o=1,\dots,O,\ i=1,\dots,I} \in \mathbb{R}_+^{OI}$.

To achieve the challenges discussed previously, manufacturers incur a variety of costs that influence their overall profitability and sustainability. Storage costs represent a significant consideration, as manufacturers must manage both raw materials and finished products. The cost of storing raw materials $\bar{c}_m^{1s}$, $m = 1, \dots, M$, depends on the total quantity procured from suppliers, including both the regular supply and any additional stock purchased in response to anticipated fluctuations in demand, that is:

$$\bar{c}_m^{1s} \equiv \bar{c}_m^{1s}\left(\sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{i=1}^{I}(x_{nrmi}+y_{nrmi})\right) = \bar{c}_m^{1s}(x_m^1, y_m^1), \quad \forall m=1,\dots,M,$$

where $(x_m^1, y_m^1) = (x_{nrmi}, y_{nrmi})_{n=1,\dots N,\ r=1,\dots,R,\ i=1,\dots,I} \in \mathbb{R}_+^{2NRI}$.

Similarly, finished product storage costs arise when manufacturers retain unsold goods, leading to additional expenditures for warehouse management and inventory control. We denote by $\bar{c}_m^{2s}$, $m = 1\dots,M$, such a cost and we observe that

$$\bar{c}_m^{2s} \equiv \bar{c}_m^{2s}(g_m) = \bar{c}_m^{2s}(g_m^1, g_m^2), \quad \forall m=1,\dots,M.$$

Another critical aspect is production costs, which encompass expenses related to transforming raw materials into market-ready products, denoted by $\bar{c}_m$, $m=1,\dots,M$. These costs vary depending on factors such as energy consumption, labor, and the complexity of the manufacturing process. Efficient production planning can help reduce these costs by optimizing resource utilization and minimizing waste. We observe that:

$$\bar{c}_m \equiv \bar{c}_m(g_m) = \bar{c}_m(g_m^1, g_m^2), \quad \forall m=1,\dots,M.$$

Manufacturers must also account for transportation costs, which are incurred at multiple stages of the supply chain. First, raw materials must be transported from suppliers to manufacturing facilities, with costs influenced by factors such as distance, transport mode, and fuel prices. We denote these costs by $\bar{c}_{nrmi}^t$, $n=1,\dots,N,\ r=1,\dots,R,\ m=1,\dots,M,\ i=1,\dots,I$, and we assume that

$$\bar{c}_{nrmi}^t \equiv \bar{c}_{nrmi}^t(x_{nrmi}, y_{nrmi}), \quad \forall n=1,\dots,N,\ r=1,\dots,R,\ m=1,\dots,M,\ i=1,\dots,I.$$

Once production is complete, manufacturers face additional transportation costs associated with delivering finished goods. If products are sold through retailers, manufacturers must arrange for shipments to distribution centers or retail locations, incurring distribution transport costs, denoted by $\bar{c}_{mli}^t$. We assume that:

$$\bar{c}_{mli}^t \equiv \bar{c}_{mli}^t(g_{mli}), \quad \forall m=1,\dots,M,\ l=1,\dots,L,\ i=1,\dots,I.$$

Alternatively, if they engage in direct-to-customer sales, manufacturers bear the cost of delivering products directly to end consumers, often using different logistics providers or fulfillment strategies. We denote these costs by $\bar{c}_{moi}^T$ and we assume that:

$$\bar{c}_{moi}^T \equiv \bar{c}_{moi}^T(g_{moi}), \quad \forall m=1,\dots,M,\ o=1,\dots,O,\ i=1,\dots,I.$$

For simplicity, explicit warehousing costs for raw materials and finished products are not modeled separately. However, the overproduction and underproduction penalty functions capture the economic impact of temporary inventory adjustments. This abstraction allows focusing on strategic interactions among agents without compromising the general equilibrium structure of the FPVN.

Beyond financial costs, manufacturers must also consider environmental sustainability constraints, particularly regarding emissions generated throughout the production and distribution processes. Regulatory policies often impose strict emission limits, requiring manufacturers to adopt cleaner technologies, optimize transportation routes, and improve energy efficiency in their facilities. Let $\bar{e}_m \equiv \bar{e}_m(g_m) = \bar{e}_m(g_m^1, g_m^2)$, $\bar{e}_{mli}^1 \equiv \bar{e}_{mli}^1(g_{mli})$, $\bar{e}_{moi}^2 \equiv \bar{e}_{moi}^2(g_{moi})$ be the environmental emissions associated with the production of the end product and transporting it to a generic retailer l and to a generic demand market o using transport mode i, respectively, for any $m=1,\dots,M$, $l=1,\dots,L$, $o=1,\dots,O$ and $i=1,\dots,I$.

We assume that the above cost and emission functions are convex and continuously differentiable with respect to their own variables.

A crucial challenge that manufacturers must address is demand uncertainty, which arises in both intermediary distribution channels and direct demand markets. The demand from retailers, who act as intermediaries between manufacturers and end customers, is subject to fluctuations due to market trends, pricing strategies, seasonality, and competitive dynamics. Each retailer forecasts its procurement needs based on expected demand, but these estimates may deviate from actual market demand, leading to situations of overstocking or shortages within the supply chain.

Similarly, in direct customer markets, where manufacturers sell finished products without relying on intermediaries, demand uncertainty is even more pronounced. Consumer behavior, market trends, competitive forces, and macroeconomic conditions all contribute to high demand variability. Unlike retailer-driven distribution, where manufacturers benefit from aggregated orders, direct sales expose manufacturers to a more fragmented and unpredictable demand structure, requiring more refined forecasting and inventory management strategies (see [51]).

From a mathematical perspective, both retailer demand and direct market demand are modeled as stochastic variables. We note that the "projected" demand for manufacturer m, $m = 1, \ldots, M$, at retailer l, $l = 1, \ldots, L$, is $\sum_{i=1}^{I} g_{mli}$. Let d_{ml} denote the actual demand at retailer l, $l = 1, \ldots, L$, for manufacturer m, $m = 1, \ldots, M$, which is a random variable with probability density function given by f_{ml}. Let P_{ml} be the probability distribution function of d_{ml} that is:

$$P_{ml}(D_{ml}) = P(d_{ml} \le D_{ml}) = \int_0^{D_{ml}} f_{ml}(t)d(t).$$

The total stochastic demand that manufacturer m must satisfy from all retailers is $d_m^1 = \sum_{l=1}^{L} d_{ml}$.

Similarly, the "projected" demand at demand market o, $o = 1, \ldots, O$, is $\sum_{i=1}^{I} g_{moi}$. Let d_{mo} denote the actual demand at demand market o, $o = 1, \ldots, O$, for manufacturer m, $m = 1, \ldots, M$, which is a random variable with probability density function given by f_{mo}. Let P_{mo} be the probability distribution function of d_{mo} that is:

$$P_{mo}(D_{mo}) = P(d_{mo} \le D_{mo}) = \int_0^{D_{mo}} f_{mo}(t)d(t).$$

The total stochastic demand that manufacturer m must satisfy from all demand markets is $d_m^2 = \sum_{o=1}^{O} d_{mo}$.

Given this inherent uncertainty, manufacturers may experience overproduction—when the quantity produced exceeds actual demand—or underproduction, where production falls short of market needs. These inefficiencies result in additional costs related to storage of unsold inventory or lost sales opportunities, both of which negatively impact profitability. We denote the overproduction and the underproduction incurred by manufacturer m, $m = 1, \ldots, M$, related to the supply of the product to all retailers, respectively, by

$$\Delta_m^{1,+} = \max\left\{0, \sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli} - d_m^1\right\}, \quad \forall m = 1, \ldots, M; \tag{17}$$

$$\Delta_m^{1,-} = \max\left\{0, d_m^1 - \sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli}\right\}, \quad \forall m = 1, \ldots, M. \tag{18}$$

Similarly, we denote the overproduction and the underproduction incurred by manufacturer m, $m = 1, \ldots, M$, related to the supply of the product to all clients, respectively, by

$$\Delta_m^{2,+} = \max\left\{0, \sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi} - d_m^2\right\}, \tag{19}$$

$$\Delta_m^{2,-} = \max\left\{0, d_m^2 - \sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi}\right\}. \tag{20}$$

We note that:

- when $\Delta_m^{1,+} > 0$ and $\Delta_m^{2,+} > 0$, it indicates that the manufacturer has produced more than the actual demand from retailers and demand markets, respectively. From a managerial point of view, this means that the total quantity of finished goods allocated to retailers and demand markets exceeds the realized demand. This situation leads to overproduction, where excess inventory accumulates, increasing storage costs and capital immobilization. Excess stock may result in additional expenses related to warehousing, insurance, and potential obsolescence if the products have a limited shelf life. In extreme cases, manufacturers might be forced to offer discounts or dispose of surplus products, further impacting profitability.

- when $\Delta_m^{1,-} > 0$ and $\Delta_m^{2,-} > 0$, it indicates that the manufacturer has produced less than the actual demand from retailers and demand markets, respectively. From a managerial point of view, this means that the total quantity of finished goods allocated to retailers is insufficient to meet the realized demand. This situation leads to underproduction, where the manufacturer is unable to fulfill all retailer and demand market orders, resulting in lost sales opportunities and customer dissatisfaction. Retailers may seek alternative suppliers to compensate for the shortage, potentially weakening the manufacturer's competitive position in the market. Moreover, unmet demand can lead to penalties or contractual breaches, further impacting the manufacturer's profitability and reputation.

In this context demand is inherently uncertain. At the same time, manufacturers must respond efficiently to market fluctuations to avoid disruptions and inefficiencies. Therefore, it is crucial for manufacturers to implement robust demand estimation strategies to ensure timely and optimal production planning, minimizing the risks of overproduction and underproduction while maintaining competitiveness.

The expected values of $\Delta_m^{1,+}$, $\Delta_m^{2,+}$, $\Delta_m^{1,-}$ and $\Delta_m^{2,-}$, respectively, are:

$$\mathbb{E}[\Delta_m^{1,+}] = \int_{\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli}}^{\infty} \left(\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli} - t\right) f_m^1(t)d(t), \quad \forall m = 1, \dots, M, \tag{21}$$

$$\mathbb{E}[\Delta_m^{1,-}] = \int_0^{\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli}} \left(t - \sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli}\right) f_m^1(t)d(t), \quad \forall m = 1, \dots, M, \tag{22}$$

$$\mathbb{E}[\Delta_m^{2,+}] = \int_{\sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi}}^{\infty} \left(\sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi} - t\right) f_m^2(t)d(t), \quad \forall m = 1, \dots, M, \tag{23}$$

$$\mathbb{E}[\Delta_m^{2,-}] = \int_0^{\sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi}} \left(t - \sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi}\right) f_m^2(t)d(t), \quad \forall m = 1, \dots, M, \tag{24}$$

where

$$f_m^1(z) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{l=1}^{L} f_{ml}(t_l)\delta\left(z - \sum_{t=1}^{L} t_l\right) dt_1 \dots dt_L \quad \forall m = 1, \dots, M,$$

$$f_m^2(z) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{o=1}^{O} f_{mo}(t_o)\delta\left(z - \sum_{o=1}^{O} t_o\right) dt_1 \dots dt_O \quad \forall m = 1, \dots, M,$$

are the probability density functions of the total demand, that can be expressed as the convolution of the probability density functions of the individual demand components, considering that demand from different sources is independent. These formulations allow us to derive the expected values of overproduction and underproduction, which play a crucial role in determining the penalties applied within the optimization model. By incorporating convolution-based probability distributions, the model effectively captures the stochastic nature of demand and provides a robust framework for decision-making under uncertainty.

To mitigate the risks associated to over- and underproduction, penalty mechanisms are introduced, discouraging excessive production while ensuring sufficient supply to meet demand. We denote by

$$\Delta_m^+ = \Delta_m^{1,+} + \Delta_m^{2,+}, \quad \forall m = 1,\dots,M, \tag{25}$$

$$\Delta_m^- = \Delta_m^{1,-} + \Delta_m^{2,-}, \quad \forall m = 1,\dots,M, \tag{26}$$

the overall overproduction and underproduction and by δ_m^+ and δ_m^-, $m = 1,\dots,M$, the penalties associated with the overproduction and underproduction, respectively. Hence, the expected total penalty for manufacturer m, $m = 1,\dots,M$, is:

$$\mathbb{E}[\delta_m^+ \Delta_m^+ + \delta_m^- \Delta_m^-] = \delta_m^+ \mathbb{E}[\Delta_m^+] + \delta_m^- \mathbb{E}[\Delta_m^-]. \tag{27}$$

Decision variables, cost and emissions functions for manufacturer m, $m = 1,\dots,M$, are listed in Table A.3 and Table A.4 in Appendix A.

Manufacturers, like the Raw Material Suppliers, are modeled as rational and intelligent agents engaging in a noncooperative competition within a GNE framework. Each manufacturer independently optimizes its decision variables—production levels, transportation strategies, and inventory management—to maximize its expected profit while minimizing costs associated with production, transportation, and storage and penalties associated with over- and underproduction. However, their strategic interactions are constrained by a shared environmental restriction: total emissions across all manufacturer must not exceed a common regulatory threshold. As at the Raw Material Suppliers tier, this introduces a coupling constraint, making the equilibrium problem more complex, since each manufacturer's strategies depend not only on market conditions but also on the emission levels resulting from the decisions of all other manufacturers in the network.

Each manufacturer m, $m = 1,\dots,M$, thus, faces the following constrained optimization problem, where it aims to maximize its expected utility:

$$\begin{aligned}
\max E(U_m) = {} & \sum_{o=1}^{O}\sum_{i=1}^{I} \bar{p}_{moi}^{2*} g_{moi} + \sum_{l=1}^{L}\sum_{i=1}^{I} \bar{\rho}_{mli}^{2*} g_{mli} - \bar{c}_m(g_m^1, g_m^2) - \sum_{l=1}^{L}\sum_{i=1}^{I} \bar{c}_{mli}^{t}(g_{mli}) - \sum_{o=1}^{O}\sum_{i=1}^{I} \bar{c}_{moi}^{T}(g_{moi}) \\
& - \sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{i=1}^{I} p_{nrmi}^{1*} x_{nrmi} - \sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{i=1}^{I} \rho_{nrmi}^{1*} y_{nrmi} - \bar{c}_m^{1s}(x_m, y_m) - \bar{c}_m^{2s}(g_m^1, g_m^2) - \sum_{n=1}^{N}\sum_{i=1}^{I} \bar{c}_{nmi}^{t}(x_{nmi}, y_{nmi}) \\
& - \bar{\alpha}_m \left[\bar{e}_m(g_m^1, g_m^2) + \sum_{l=1}^{L}\sum_{i=1}^{I} \bar{e}_{mli}^1(g_{mli}) + \sum_{o=1}^{O} \bar{e}_{moi}^2(g_{moi}) \right] - \delta_m^- \mathbb{E}[\Delta_m^-] - \delta_m^+ \mathbb{E}[\Delta_m^+],
\end{aligned} \tag{28}$$

subject to constraints:

$$\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli} + \sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi} \le \bar{Q}_m, \tag{29}$$

$$\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli} + \sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi} + \underline{g}_m \le \bar{S}_m^1 \tag{30}$$

$$\sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{i=1}^{I} x_{nrmi} + \sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{i=1}^{I} y_{nrmi} + \underline{x}_{mr} \le \overline{S}_m^2 \tag{31}$$

$$\sum_{i=1}^{I} x_{nrmi} + \sum_{i=1}^{I} y_{nrmi} + \underline{x}_{mr} \ge \beta_{nrm} \cdot \left(\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli} + \sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi} \right), \quad \forall n, r, m \tag{32}$$

$$x_{nmi}, y_{nmi}, g_m, g_{mli}, g_{moi} \ge 0, \quad \forall l,\ o,\ i \tag{33}$$

$$\sum_{m=1}^{M} \left[\bar{e}_m(g_m^1, g_m^2) + \sum_{l=1}^{L}\sum_{i=1}^{I} \bar{e}_{mli}^1(g_{mli}) + \sum_{o=1}^{O} \bar{e}_{moi}^2(g_{moi}) \right] \le \bar{\varepsilon}. \tag{34}$$

Expression (28) represents a weighted objective function that balances total expected profit—derived from end product sales to retailers and demand markets minus production, storage and transportation costs, overall penalties due to over- and underproduction as well as amount paid for the purchase of raw materials from all raw material suppliers—and total environmental emissions. In this formulation, $\bar{p}^{2*}_{moi}$ and $\bar{\rho}^{2*}_{mli}$, respectively, represent the unit price obtained by the sale of a unit of end product to a demand market o, $o = 1, ..., O$ with the shipment method i, $i = 1, ..., I$, and the unit price obtained by the sale of a unit of end product to a retailer l, $l = 1, \dots, L$ with the shipment method i, $i = 1, ..., I$. Moreover, $\bar{\alpha}_m > 0$ denotes a parameter that allows us to express environmental emissions in terms of costs and enables distinct manufacturers to have different emissions in terms of costs.

Constraint (29) represents a production capacity for manufacturer m. It establishes that the quantity of finished product sold to all retailers and all demand markets cannot exceed the operation condition $\bar{\mathcal{Q}}_m$, which reflects the manufacturer's physical and technological production capabilities.

The total storage capacity for finished products of manufacturer m is represented by the parameter $\bar{S}^1_m$. Constraint (30) ensures that the sum of the initial stock of finished product quantities ($\underline{g}_m$) and the product that the manufacturer supplies to customers and retailers (hence, the quantity produced) does not exceed the storage capacity.

Similarly, constraint (31) ensures that the sum of the initial stock of raw material of type r ($\underline{x}_{mr}$) and the raw material amount that the manufacturer purchases from all suppliers does not exceed the storage capacity for raw material $\overline{S}^2_m$.

Constraint (32) ensures that the quantity of raw material of type r purchased from each supplier to produce the finished product is sufficient to satisfy the production demand. Specifically, the constraint states that the raw material of type r purchased from each manufacturer, plus any initial stock of such a type of raw material, i.e. $\underline{x}_{mr} \geq 0$, must be at least equal to the quantity of raw material of type r needed to produce the quantity of product requested, expressed by the parameter β_{nmr}.

Constraint (33) defines the domain of manufacturer decision variables.

Finally, constraint (34) accounts for the total environmental impact of manufacturers by aggregating emissions from production, transportation, and storage. Since emissions contribute to a shared regulatory limit $\bar{\varepsilon}$, this constraint introduces interdependence among suppliers, making it a coupling constraint in the Generalized Nash Equilibrium framework, as in the Raw Material Suppliers level.
We define the set for each manufacturer m, $m = 1, \dots, M$ as:

$$\bar{K}^m = \{\bar{\psi}_m \equiv (g^1_m, g^2_m, x^1_m, y^1_m) \in \mathbb{R}^{LI+OI+2NRI}_+ | (29) - (33) \text{ hold}\}$$

and we denote by $\bar{\mathcal{S}}$ the set of shared constraints, namely:

$$\bar{\mathcal{S}} = \{\bar{\psi} \equiv (g^1, g^2, x^1, y^1) \in \mathbb{R}^{M(LI+OI+2NRI)}_+ | (34) \text{ holds}\}.$$

Finally, we define the feasible set $\bar{K} = \prod_{m=1}^{M} \bar{K}^m$ and $\bar{\mathbb{K}} = \bar{K} \cap \bar{\mathcal{S}}$.

We refer to objective function (28) as the expected utility $E(U_m)$, for $m = 1, \dots, M$. Assuming that (as stated before) the cost and emission functions for each raw material supplier are convex and continuously differentiable, we can state:

Definition 2.3 (Generalized Nash Equilibrium for Manufacturers) *A strategy profile $\bar{\psi}^* \in \bar{\mathbb{K}}$ is a Generalized Nash Equilibrium if, for any Manufacturer m, $m = 1, \dots, M$,*

$$\begin{aligned} E(U_m(g^{1*}_m, g^{2*}_m, x^{1*}_m, y^{1*}_m, g^{1*}_{-m}, g^{2*}_{-m}, x^{1*}_{-m}, y^{1*}_{-m})) \\ \geq E(U_m(g^1_m, g^2_m, x^1_m, y^1_m, g^{1*}_{-m}, g^{2*}_{-m}, x^{1*}_{-m}, y^{1*}_{-m})) \\ \forall \bar{\psi}_m \in \bar{K}^m, \quad \forall \bar{\psi} \in \bar{\mathcal{S}}, \end{aligned} \tag{35}$$

where

$$g^{1*}_{-m} = (g^{1*}_1, \dots, g^{1*}_{m-1}, g^{1*}_{m+1}, \dots, g^{1*}_M), \ g^{2*}_{-m} = (g^{2*}_1, \dots, g^{2*}_{m-1}, g^{2*}_{m+1}, \dots, g^{2*}_M),$$

$$x^{1*}_{-m} = (x^{1*}_1, \dots, x^{1*}_{m-1}, x^{1*}_{m+1}, \dots, x^{1*}_M), \ y^{1*}_{-m} = (y^{1*}_1, \dots, y^{1*}_{m-1}, y^{1*}_{m+1}, \dots, y^{1*}_M).$$

Each manufacturer wishes to maximize its expected utility. The above definition states that no manufacturer, given the circumstances and the strategies of the other manufacturers, at equilibrium, is willing unilaterally to change its vector of strategies because it may end up with a lower expected utility. We observe that the expected utility of each manufacturer depends not only on its own decisions but also on the strategies of other manufacturers. In addition, their feasible sets are interconnected due to shared constraints on environmental emissions. Therefore, as in the Raw Material Suppliers level, we have a Generalized Nash Equilibrium model.

For simplicity, we now present some additional preliminaries. For each manufacturer m, we get:

$$\frac{\partial \mathbb{E}[\Delta_m^+]}{\partial g_{mli}} = P_m\left(\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli}\right), \quad \forall l = 1, \dots, L, \; i = 1, \dots, I, \tag{36}$$

$$\frac{\partial \mathbb{E}[\Delta_m^+]}{\partial g_{moi}} = P_m\left(\sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi}\right), \quad \forall o = 1, \dots, O, \; i = 1, \dots, I, \tag{37}$$

$$\frac{\partial \mathbb{E}[\Delta_m^-]}{\partial g_{mli}} = P_m\left(\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli}\right) - 1, \quad \forall l = 1, \dots, L, \; i = 1, \dots, I, \tag{38}$$

$$\frac{\partial \mathbb{E}[\Delta_m^-]}{\partial g_{moi}} = P_m\left(\sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi}\right) - 1, \quad \forall o = 1, \dots, O, \; i = 1, \dots, I, \tag{39}$$

and

$$\frac{\partial}{\partial g_{mli}}\left(\delta_m^+\mathbb{E}[\Delta_m^+] + \delta_m^-\mathbb{E}[\Delta_m^-]\right) = (\delta_m^+ + \delta_m^-) f_m^1\left(\sum_{l=1}^{L}\sum_{i=1}^{I} g_{mli}\right), \qquad \forall l = 1, \dots, L, \; i = 1, \dots, I, \tag{40}$$

$$\frac{\partial}{\partial g_{moi}}\left(\delta_m^+\mathbb{E}[\Delta_m^+] + \delta_m^-\mathbb{E}[\Delta_m^-]\right) = (\delta_m^+ + \delta_m^-) f_m^2\left(\sum_{o=1}^{O}\sum_{i=1}^{I} g_{moi}\right), \qquad \forall o = 1, \dots, O, \; i = 1, \dots, I. \tag{41}$$

Hence, it is obvious that $\delta_m^+\mathbb{E}[\Delta_m^+] + \delta_m^-\mathbb{E}[\Delta_m^-]$ is convex.

At the Raw Material Suppliers level, we consider the class of Variational Equilibria.

Definition 2.4 (Variational Equilibrium) *A strategy vector $\bar{\psi}^* \in \bar{\mathbb{K}}$ is a Variational Equilibrium of the above Generalized Nash Equilibrium problem if $\bar{\psi}^* \in \bar{\mathbb{K}}$ is a solution to the variational inequality:*

$$\begin{aligned}
&\sum_{l=1}^{L}\sum_{i=1}^{I}\Bigg[\frac{\partial \bar{c}_m^{2s}(g_m^{1*}, g_m^{2*})}{\partial g_{mli}} + \frac{\partial \bar{c}_{mli}^{t}(g_{mli}^*)}{\partial g_{mli}} + \bar{\alpha}_m\left(\frac{\partial \bar{e}_m(g_m^{1*}, g_m^{2*})}{\partial g_{mli}} + \frac{\partial \bar{e}_{mli}^{1}(g_{mli}^*)}{\partial g_{mli}}\right) \\
&\qquad\qquad + \delta_m^-\frac{\partial \mathbb{E}[\Delta_m^-]}{\partial g_{mli}} + \delta_m^+\frac{\partial \mathbb{E}[\Delta_m^+]}{\partial g_{mli}} - \bar{\rho}_{mli}^{2*}\Bigg] \times (g_{mli} - g_{mli}^*) \\
&+ \sum_{o=1}^{O}\sum_{i=1}^{I}\Bigg[\frac{\partial \bar{c}_m^{2s}(g_m^{1*}, g_m^{2*})}{\partial g_{moi}} + \frac{\partial \bar{c}_{moi}^{T}(g_{moi}^*)}{\partial g_{moi}} + \bar{\alpha}_m\left(\frac{\partial \bar{e}_m(g_m^{1*}, g_m^{2*})}{\partial g_{moi}} + \frac{\partial \bar{e}_{moi}^{2}(g_{moi}^*)}{\partial g_{moi}}\right) \\
&\qquad\qquad + \delta_m^-\frac{\partial \mathbb{E}[\Delta_m^-]}{\partial g_{moi}} + \delta_m^+\frac{\partial \mathbb{E}[\Delta_m^+]}{\partial g_{moi}} - \bar{p}_{moi}^{2*}\Bigg] \times (g_{moi} - g_{moi}^*) \\
&+ \sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{i=1}^{I}\left[\frac{\partial \bar{c}_m^{1s}(x_m^*, y_m^*)}{\partial x_{nrmi}} + \frac{\partial \bar{c}_{nrmi}^{t}(x_{nrmi}^*, y_{nrmi}^*)}{\partial x_{nrmi}} + p_{nrmi}^{1*}\right] \times (x_{nrmi} - x_{nrmi}^*) \\
&+ \sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{i=1}^{I}\left[\frac{\partial \bar{c}_m^{1s}(x_m^*, y_m^*)}{\partial y_{nrmi}} + \frac{\partial \bar{c}_{nrmi}^{t}(x_{nrmi}^*, y_{nrmi}^*)}{\partial y_{nrmi}} + \rho_{nrmi}^{1*}\right] \times (y_{nrmi} - y_{nrmi}^*) \geq 0, \quad \forall \bar{\psi} \in \bar{\mathbb{K}}.
\end{aligned} \tag{42}$$

For simplicity of notation, we now put variational inequality (42) into standard form, that is: determine $\bar{X}^* \in \bar{\mathcal{K}}$ such that:

$$\langle \bar{F}(\bar{X}^*), \bar{X} - \bar{X}^* \rangle \geq 0, \quad \forall \bar{X} \in \bar{\mathcal{K}}, \tag{43}$$

where $\bar{\mathcal{K}}$ is a closed and convex set.

To this purpose, we define $\bar{X} \equiv \bar{\psi}$, $\bar{F}(\bar{X}) = (\bar{F}^1(\bar{X}), \bar{F}^2(\bar{X}), \bar{F}^3(\bar{X}), \bar{F}^4(\bar{X}))$ and $\bar{\mathcal{K}} \equiv \bar{\mathbb{K}}$, where:

$$\begin{aligned}
\bar{F}^1_{mli}(\bar{X}) &= \left[\frac{\partial \bar{c}^{2s}_m(g^1_m, g^2_m)}{\partial g_{mli}} + \frac{\partial \bar{c}^t_{mli}(g_{mli})}{\partial g_{mli}} + \bar{\alpha}_m \left(\frac{\partial \bar{e}_m(g^1_m, g^2_m)}{\partial g_{mli}} + \frac{\partial \bar{e}^1_{mli}(g_{mli})}{\partial g_{mli}}\right)\right. \\
&\qquad \left. + \delta^-_m \frac{\partial \mathbb{E}[\Delta^-_m]}{\partial g_{mli}} + \delta^+_m \frac{\partial \mathbb{E}[\Delta^+_m]}{\partial g_{mli}} - \bar{\rho}^{2*}_{mli}\right], \quad \forall m, l, i; \\
\bar{F}^2_{moi}(\bar{X}) &= \left[\frac{\partial \bar{c}^{2s}_m(g^1_m, g^2_m)}{\partial g_{moi}} + \frac{\partial \bar{c}^T_{moi}(g_{moi})}{\partial g_{moi}} + \bar{\alpha}_m \left(\frac{\partial \bar{e}_m(g^1_m, g^2_m)}{\partial g_{moi}} + \frac{\partial \bar{e}^2_{moi}(g_{moi})}{\partial g_{moi}}\right)\right. \\
&\qquad \left. + \delta^-_m \frac{\partial \mathbb{E}[\Delta^-_m]}{\partial g_{moi}} + \delta^+_m \frac{\partial \mathbb{E}[\Delta^+_m]}{\partial g_{moi}} - \bar{p}^{2*}_{moi}\right], \quad \forall m, o, i; \\
\bar{F}^3_{nrmi}(\bar{X}) &= \left[\frac{\partial \bar{c}^{1s}_m(x_m, y_m)}{\partial x_{nrmi}} + \frac{\partial \bar{c}^t_{nmi}(x_{nrmi}, y_{nrmi})}{\partial x_{nrmi}} + p^{1*}_{nrmi}\right], \quad \forall n, r, m, i; \\
\bar{F}^4_{nrmi}(\bar{X}) &= \left[\frac{\partial \bar{c}^{1s}_m(x_m, y_m)}{\partial y_{nrmi}} + \frac{\partial \bar{c}^t_{nrmi}(x_{nrmi}, y_{nrmi})}{\partial y_{nrmi}} + \rho^{1*}_{nrmi}\right], \quad \forall n, r, m, i.
\end{aligned} \tag{44}$$

Under the assumptions imposed on the cost and emission functions, we can deduce that the function $\bar{F}(\bar{X})$ that defines the variational inequality (43) is continuous. Moreover, it is easy to show that the feasible set $\bar{\mathcal{K}}$ is compact (since constraints (29)-(31) and conservation law (16) hold). Therefore, the existence of a solution to variational inequality (43) is guaranteed by the classical theory of variational inequalities (see [23]).

2.3 Retailers' Behavior

Retailers act as key intermediaries in the FPVN, bridging the gap between manufacturers and the final customers at the demand markets. They are responsible for procuring finished goods from manufacturers, managing inventory levels and fulfilling customer demand while optimizing transportation and pricing strategies. Retailers operate in a competitive and decentralized environment where each entity independently maximizes its expected utility within a Generalized Nash Equilibrium GNE framework.

Retailers face demand uncertainty, requiring them to balance inventory levels to avoid both overstocking, which increases storage costs, and under stocking, which results in lost sales. To mitigate these risks, penalty mechanisms are introduced, discouraging inefficiencies and promoting alignment between procurement decisions and actual demand. In addition, transportation and distribution activities contribute to emissions, subjecting retailers to shared regulatory restrictions on environmental impact.

Their optimization problems integrate economic, operational, and environmental considerations, ensuring that decisions regarding procurement, storage, and pricing contribute to network-wide stability. The equilibrium formulation guarantees that no retailer can unilaterally improve its utility without affecting the broader FPVN, reinforcing the interdependence of all supply chain actors. By optimizing their strategies, retailers enhance both efficiency and sustainability in decentralized production networks.

Each retailer l, $l = 1, \ldots, L$, must determine the quantity of end product purchased from manufacturer m, $m = 1, \ldots, M$, using transport mode $i = 1, \ldots, I$, previously denoted as $g_{mli} \in \mathbb{R}_+$. The end product that can be sold to demand market o, $o = 1, \ldots, O$, via transport mode i is denoted by $q_{loi} \in \mathbb{R}_+$. Additionally, each retailer maintains an inventory stock s_l to buffer against sudden fluctuations in demand. The presence of inventory allows retailers to respond promptly to customer needs, mitigating the risk of stock-outs, which may lead to lost sales or customer dissatisfaction. However, excessive inventory results in additional holding costs, creating a trade-off that retailers must carefully manage. Denoted by $\underline{s}_l$ the quantity of finished product that the supplier has in stock, the following conservation law holds:

$$s_l = \underline{s}_l + \sum_{m=1}^{M} \sum_{i=1}^{I} g_{mli} - \sum_{o=1}^{O} \sum_{i=1}^{I} q_{loi}, \quad \forall l = 1, \ldots, L. \tag{45}$$

The above conservation law ensures that every unit of product entering the retailer's node (received from manufacturers and held in stock) is fully allocated, either through sales to customers or by holding it as residual stock. Therefore, s_l can be expressed in terms of the vectors $g^1_l = (g_{mli})_{m=1,\ldots,M,\ i=1,\ldots,I}$ and $q_l = (q_{loi})_{o=1,\ldots,O,\ i=1,\ldots,I}$.

Retailers incur a variety of costs that influence their overall profit and sustainability. Specifically, retailer l incurs transportation costs, $\tilde{c}^T_{mli}$, $m = 1,\dots M$, $l = 1,\dots,L$, $i = 1,\dots,I$, associated with acquiring goods from manufacturer m via transport mode i. We observe that:

$$\tilde{c}^T_{mli} \equiv \tilde{c}^T_{mli}(g_{mli}), \quad \forall m = 1,\dots M,\ l = 1,\dots,L,\ i = 1,\dots,I.$$

Moreover, retailer l faces additional transportation costs associated with delivering end product to demand market o via shipment method i, denoted by $\tilde{c}_{loi}$, and we assume that

$$\tilde{c}^t_{loi} \equiv \tilde{c}^t_{loi}(q_{loi}), \quad \forall l = 1,\dots,L,\ o = 1,\dots,O\ i = 1,\dots,I.$$

Storage costs, denoted by $\tilde{c}_l$, represent a significant consideration. These costs depend on the quantity of inventory held, that is:

$$\tilde{c}^s_l \equiv \tilde{c}^s_l(s_l) = \tilde{c}_l(g^1_l, q_l), \quad \forall l = 1,\dots,L.$$

As in the previous tiers of the network, retailers must also consider environmental sustainability constraints, particularly regarding emissions generated throughout the storage of end product and distribution processes. Regulatory policies often impose strict emission limits, requiring retailers to optimize transportation routes. Let $\tilde{e}_l \equiv \tilde{e}_l(s_l) = \tilde{e}_l(g^1_l, q_l)$ and $\tilde{e}_{loi} \equiv \tilde{e}_{loi}(q_{loi})$ be the environmental emissions associated with the storage of end product and transporting of it to a generic demand market o and using transport mode i, for any $l = 1,\dots,L$, $o = 1,\dots,O$ and $i = 1,\dots,I$.

We assume that the above cost and emission functions are convex and continuously differentiable with respect to their own variables.

Similar to the tier of manufacturers, retailers must address demand uncertainty. The demand from all demand markets is subject to fluctuations due to markets trends, pricing strategies, seasonality, and competitive dynamics. Each retailer forecasts its procurement needs based on expected demand, but these estimates may deviate from actual market demand, leading to situations of overstocking or under stocking.

We note that the "projected" demand for retailer l, $l = 1,\dots,L$, at demand market o, $o = 1,\dots,O$, is $\sum_{i=1}^{I} q_{loi}$. Let d_{lo} denote the actual demand at demand market o, $o = 1,\dots,O$, for retailer l, $l = 1,\dots,L$, which is a random variable with probability density function given by f_{lo}. Let P_{lo} be the probability distribution function of d_{lo} that is:

$$P_{lo}(D_{lo}) = P(d_{lo} \leq D_{lo}) = \int_0^{D_{lo}} f_{lo}(t)d(t).$$

The total stochastic demand that retailer l must satisfy from all demand markets is $d_l = \sum_{o=1}^{O} d_{lo}$.

Given the inherent uncertainty, retailers may experience overstocking—when the quantity of end product owned by retailers exceeds actual demand—or under stocking, where the total stock falls short of market needs. The inefficies result in additional costs related to storage of unsold inventory or lost sales opportunities. We denote the overstocking and the under stocking incurred by retailer l, $l = 1,\dots,L$, related to the supply of the product to all demand markets, respectively, by:

$$\Delta^+_l = \max\left\{0, \sum_{o=1}^{O}\sum_{i=1}^{I} q_{loi} + s_l - d_l\right\}, \quad \forall l = 1,\dots,L; \tag{46}$$

$$\Delta^-_l = \max\left\{0, d_l - \left(\sum_{o=1}^{O}\sum_{i=1}^{I} q_{loi} + s_l\right)\right\}, \quad \forall l = 1,\dots,L. \tag{47}$$

Overstocking occurs when the quantity of product sold to customers and the quantity of product in stock exceeds actual customer demand while under stocking occurs when the quantity of product sold to customers and the quantity of product in stock is less than the random demand of customers. Specifically:

- when $\Delta_l^+ > 0$, this means that what the retailer sends to customers plus the stocks in the warehouse exceed the actual demand, generating a surplus of product in the warehouse;
- when $\Delta_l^- > 0$, this means that what the retailer sends to customers plus the inventory in the warehouse is insufficient to cover the actual demand, leading to customer dissatisfaction or loss of sales opportunities.

It is crucial for retailers to implement robust demand estimation strategies to ensure timely and optimal distribution planning, minimizing the risks of overstocking and under stocking while maintaining competitiveness. The expected value of Δ_l^+ and Δ_l^- are:

$$\mathbb{E}[\Delta_l^+] = \int_{\sum_{o=1}^{O}\sum_{i=1}^{I} q_{loi} + s_l}^{\infty} \left(\sum_{o=1}^{O}\sum_{i=1}^{I} q_{loi} + s_l - t\right) f_l(t)d(t), \ \forall l = 1, \dots, L, \tag{48}$$

$$\mathbb{E}[\Delta_l^-] = \int_0^{\sum_{o=1}^{O}\sum_{i=1}^{I} q_{loi} + s_l} \left[t - \left(\sum_{o=1}^{O}\sum_{i=1}^{I} q_{loi} + s_l\right)\right] f_l(t)d(t), \forall l = 1, \dots, L, \tag{49}$$

where

$$f_l(z) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{o=1}^{O} f_{lo}(t_o)\delta\left(z - \sum_{o=1}^{O} t_o\right) dt_1 \dots d_{t_O} \tag{50}$$

is the probability density functions of the total demand, that can be expressed as the convolution of the probability density functions of the individual demand components, considering that demand from different sources is independent.

To align procurement and sales with actual demand, retailers face penalties for inventory mismanagement. We denote by δ_l^+ and δ_l^-, $l = 1, \dots, L$, the penalties associated with the overstocking and under stocking, respectively. Hence, the expected total penalty for retailer l is:

$$\mathbb{E}[\delta_l^+\Delta_l^+ + \delta_l^-\Delta_l^-] = \delta_l^+\mathbb{E}[\Delta_l^+] + \delta_l^-\mathbb{E}[\Delta_l^-], \quad \forall l = 1, \dots, L. \tag{51}$$

Decision variables, cost and emissions function for retailer l, $l = 1, \dots, L$, are listed in Table A.5 and Table A.6 in Appendix A.

Each retailer l, $l = 1, \dots, L$, operates as an independent decision-maker in a decentralized and competitive environment, where it optimizes procurement, inventory management, and sales strategies while accounting for uncertain customer demand, transportation costs, and environmental constraints. The retailer's goal is to maximize expected utility by balancing revenues from sales against procurement, storage, and distribution costs, while minimizing penalties associated with inventory misalignment. The optimization problem is structured as follows:

$$\begin{aligned}\max E(U_l) = &\sum_{o=1}^{O}\sum_{i=1}^{I} \tilde{p}_{loi}^{3*} q_{loi} - \sum_{o=1}^{O}\sum_{i=1}^{I} \tilde{c}_{loi}^t(q_{loi}) - \sum_{m=1}^{M}\sum_{i=1}^{I} \tilde{c}_{mli}^T(g_{mli}) - \sum_{m=1}^{M}\sum_{i=1}^{I} \bar{\rho}_{mli}^{2*} g_{mli} \\ &\tilde{c}_l^s(g_l^1, q_l) - \tilde{\alpha}_l \left[\tilde{e}_l(g_l^1, q_l) + \sum_{o=1}^{O}\sum_{i=1}^{I} \tilde{e}_{loi}(q_{loi})\right] - \tilde{\delta}_l^-\mathbb{E}[\Delta_l^-] - \tilde{\delta}_l^+\mathbb{E}[\Delta_l^+]\end{aligned} \tag{52}$$

subject to constraints:

$$\sum_{m=1}^{M}\sum_{i=1}^{I} g_{mli} + \underline{s}_l \leq \tilde{\mathcal{Q}}_l, \tag{53}$$

$$g_{mli}, q_{loi} \geq 0, \quad \forall m = 1, \dots, M, \ o = 1, \dots, O, \ i = 1, \dots, I, \tag{54}$$

$$\sum_{l=1}^{L}\left(\tilde{e}_l(g_l^1, q_l) + \sum_{o=1}^{O}\sum_{i=1}^{I} \tilde{e}_{loi}(q_{loi})\right) \leq \tilde{\varepsilon}. \tag{55}$$

Expression (52) represents a weighted objective function that balances total expected profit derived from end product sales to demand markets minus storage and transportation costs, overall penalties due to over- and under stocking as well as amount paid for the purchase of end products from all manufacturers—and total environmental emissions. In this formulation, $\tilde{p}^{3*}_{loi}$ represents the unit revenue obtained by the sale of a unit of end product to demand market o, $o = 1, ..., O$ with the shipment method i, $i = 1, ..., I$. Moreover, $\tilde{\alpha}_l > 0$ denotes a parameter that allows us to express environmental emissions in terms of costs and enables distinct retailers to have different emissions in terms of costs.

Constraint (53) represents a storage capacity constraint for purchased end product from all manufacturers.

Constraint (54) represents the domain of the decision variables of retailer l.

Constraint (55) accounts for the total environmental impact of retailers by aggregating emissions from transportation and storage. Since emissions contribute to a shared regulatory limit $\tilde{\varepsilon}$, this constraint introduces interdependence among retailers, making it a coupling constraint in the Generalized Nash Equilibrium framework, as in the previous tiers of the network.

We define the set for each retailer l, $l = 1, \ldots, L$, as:

$$\tilde{K}^l = \{\tilde{\psi}_l \equiv (g^1_l, q_l) \in \mathbb{R}^{MI+OI}_+ | (53) \text{ and } (54) \text{ hold}\}$$

and we denote by $\tilde{\mathcal{S}}$ the set of shared constraints, namely

$$\tilde{\mathcal{S}} = \{\tilde{\psi} \equiv (g^1, q, s) \in \mathbb{R}^{L(MI+OI)}_+ | (55) \text{ holds}\}.$$

Finally, we define the feasible set $\tilde{K} = \prod_{l=1}^{l} \tilde{K}^l$ and $\tilde{\mathbb{K}} = \tilde{K} \cap \tilde{\mathcal{S}}$.

We refer to objective function (52) as the expected utility $E(U_l)$, for $l = 1, \ldots, L$. Assuming, as stated before, that the cost and emission functions for each raw material supplier are convex and continuously differentiable, we can state:

Definition 2.5 (Generalized Nash Equilibrium for Retailers) *A strategy profile $\tilde{\psi}^* \in \tilde{\mathbb{K}}$ is a Generalized Nash Equilibrium if, for any Retailer l, $l = 1, \ldots, L$,*

$$E(U_l(g^{1*}_l, q^*_l, g^{1*}_{-l}, q^*_{-l})) \geq E(U_l(g^1_l, q_l, g^{1*}_{-l}, q^*_{-l}))$$

$$\forall \tilde{\psi}_l \in \tilde{K}^l, \quad \forall \tilde{\psi} \in \tilde{\mathcal{S}}, \tag{56}$$

where

$$g^{1*}_{-l} = (g^{1*}_1, \ldots, g^{1*}_{l-1}, g^{1*}_{l+1}, \ldots, g^{1*}_L), \; q^*_{-l} = (q^*_1, \ldots, q^*_{l-1}, q^*_{l+1}, \ldots, q^*_L).$$

Each retailer wishes to maximize its expected utility. The above definition states that no retailer, given the circumstances and the strategies of the other retailers, at equilibrium, is willing unilaterally to change its vector of strategies, because it may end up with a lower expected utility. We observe that the expected utility of each retailer depends not only on its own decisions, but also on the strategies of other retailers. Also, their feasible sets are interconnected because of the shared constraints on environmental emissions. Therefore, as in the upper levels of the network, we have a Generalized Nash Equilibrium model.

For simplicity, we now present some additional preliminaries. For each retailer l,

$$\frac{\partial E[\Delta^+_l]}{\partial q_{loi}} = P_l \left(\sum_{o=1}^{O} \sum_{i=1}^{I} q_{loi} + s_l \right), \quad \forall o = 1, \ldots, O, \; i = 1, \ldots, I, \tag{57}$$

$$\frac{\partial \mathbb{E}[\Delta^+_l]}{\partial g_{mli}} = P_l \left(\sum_{o=1}^{O} \sum_{i=1}^{I} q_{loi} + s_l \right), \quad \forall m = 1, \ldots, M, \; i = 1, \ldots, I, \tag{58}$$

$$\frac{\partial \mathbb{E}[\Delta^-_l]}{\partial q_{loi}} = P_l \left(\sum_{o=1}^{O} \sum_{i=1}^{I} q_{loi} + s_l \right) - 1, \quad \forall o = 1, \ldots, O, \; i = 1, \ldots, I, \tag{59}$$

$$\frac{\partial \mathbb{E}[\Delta^-_l]}{\partial g_{mli}} = P_l \left(\sum_{o=1}^{O} \sum_{i=1}^{I} q_{loi} + s_l \right) - 1, \quad \forall m = 1, \ldots, M, \; i = 1, \ldots, I, \tag{60}$$

and

$$\frac{\partial}{\partial q_{loi}}\left(\delta_l^+\mathbb{E}[\Delta_l^+]+\delta_l^-\mathbb{E}[\Delta_l^-]\right)=(\delta_l^++\delta_l^-)f_l\left(\sum_{o=1}^{O}\sum_{i=1}^{I}q_{loi}+s_l\right),\qquad \forall o=1,\ldots,O,\ i=1,\ldots,I. \tag{61}$$

$$\frac{\partial}{\partial g_{mli}}\left(\delta_l^+\mathbb{E}[\Delta_l^+]+\delta_l^-\mathbb{E}[\Delta_l^-]\right)=(\delta_l^++\delta_l^-)f_l\left(\sum_{o=1}^{O}\sum_{i=1}^{I}q_{loi}+s_l\right),\qquad \forall m=1,\ldots,M,\ i=1,\ldots,I, \tag{62}$$

Hence, it is obvious that $\delta_l^+\mathbb{E}[\Delta_l^+]+\delta_l^-\mathbb{E}[\Delta_l^-]$ is convex.

As done for the Raw Material Supplier and Manufacturer tiers, we consider the class of Variational Equilibria.

Definition 2.6 (Variational Equilibrium) *A strategy vector $\tilde{\psi}^*\in\tilde{\mathbb{K}}$ is a Variational Equilibrium of the above Generalized Nash Equilibrium problem if $\tilde{\psi}^*\in\tilde{\mathbb{K}}$ is a solution to the variational inequality:*

$$\begin{aligned}&\sum_{l=1}^{L}\sum_{i=1}^{I}\left[\frac{\partial\tilde{c}_{mli}^T(g_{mli}^*)}{\partial g_{mli}}+\bar{\rho}_{mli}^{2*}+\frac{\partial\tilde{c}_l^s(g_l^{1*},q_l^*)}{\partial g_{mli}}+\tilde{\alpha}_l\frac{\partial\tilde{e}_l(g_l^{1*},q_l^*)}{\partial g_{mli}}+\tilde{\delta}_l^-\frac{\partial\mathbb{E}[\Delta_l^-]}{\partial g_{mli}}+\tilde{\delta}_l^+\frac{\partial\mathbb{E}[\Delta_l^+]}{\partial g_{mli}}\right]\times(g_{mli}-g_{mli}^*)\\&+\sum_{o=1}^{O}\sum_{i=1}^{I}\left[\frac{\partial\tilde{c}_{loi}^t(q_{loi}^*)}{\partial q_{loi}}+\frac{\partial\tilde{c}_l^s(g_l^{1*},q_l^*)}{\partial q_{loi}}+\tilde{\alpha}_l\left(\frac{\partial\tilde{e}_{loi}(q_{loi}^*)}{\partial q_{loi}}+\frac{\partial\tilde{e}_l(g_l^{1*},q_l^*)}{\partial q_{loi}}\right)\right.\\&\qquad\left.+\tilde{\delta}_l^-\frac{\partial\mathbb{E}[\Delta_l^-]}{\partial q_{loi}}+\tilde{\delta}_l^+\frac{\partial\mathbb{E}[\Delta_l^+]}{\partial q_{loi}}-\tilde{p}_{loi}^{3*}\right]\times(q_{loi}-q_{loi}^*)\geq0,\quad\forall\tilde{\psi}\in\tilde{\mathbb{K}}.\end{aligned} \tag{63}$$

We now put variational inequality (63) into standard form, that is: determine $\tilde{X}^*\in\tilde{\mathbb{K}}$ such that:

$$\langle\tilde{F}(\tilde{X}^*),\tilde{X}-\tilde{X}^*\rangle\geq0,\quad\forall\tilde{X}\in\tilde{\mathcal{K}}, \tag{64}$$

where $\tilde{\mathcal{K}}$ is a closed and convex set.

To this purpose, we define $\tilde{X}\equiv\tilde{\psi}$, $\tilde{F}=(\tilde{F}^1(\tilde{X}),\tilde{F}^2(\tilde{X}))$ and $\tilde{\mathcal{K}}\equiv\tilde{\mathbb{K}}$, where:

$$\begin{aligned}&\tilde{F}_{mli}^1(\tilde{X})=\left[\frac{\partial\tilde{c}_{mli}^T(g_{mli})}{\partial g_{mli}}+\bar{\rho}_{mli}^{2*}+\frac{\partial\tilde{c}_l^s(g_l^1,q_l)}{\partial g_{mli}}+\tilde{\alpha}_l\frac{\partial\tilde{e}_l(g_l^1,q_l)}{\partial g_{mli}}+\tilde{\delta}_l^-\frac{\partial\mathbb{E}[\Delta_l^-]}{\partial g_{mli}}+\tilde{\delta}_l^+\frac{\partial\mathbb{E}[\Delta_l^+]}{\partial g_{mli}}\right],\quad\forall m,l,i;\\&\tilde{F}_{loi}^2(\tilde{X})=\left[\frac{\partial\tilde{c}_{loi}^t(q_{loi})}{\partial q_{loi}}+\frac{\partial\tilde{c}_l^s(g_l^1,q_l)}{\partial q_{loi}}+\tilde{\alpha}_l\left(\frac{\partial\tilde{e}_{loi}(q_{loi})}{\partial q_{loi}}+\frac{\partial\tilde{e}_l(g_l^1,q_l)}{\partial q_{loi}}\right)\right.\\&\qquad\left.+\tilde{\delta}_l^-\frac{\partial\mathbb{E}[\Delta_l^-]}{\partial q_{loi}}+\tilde{\delta}_l^+\frac{\partial\mathbb{E}[\Delta_l^+]}{\partial q_{loi}}-\tilde{p}_{loi}^{3*}\right],\quad\forall l,o,i.\end{aligned} \tag{65}$$

Under the imposed assumptions on the cost and emission functions, we can deduce that the function $\tilde{F}(\tilde{X})$ defining variational inequality (64) is continuous. Moreover, it is easy to show that the feasible set $\tilde{\mathcal{K}}$ is compact (since constraints (53) and conservation law (45) hold). Therefore, the existence of a solution to variational inequality (64) is guaranteed by the classical theory of variational inequalities (see [23]).

2.4 Behavior at the Demand Markets

At the lowest tier of the network, the consumers at the demand markets, as previously mentioned, can decide to buy the finished product both directly from the manufacturers and/or from the retailers. Their consumption decisions depend on the price charged for the products by the manufacturers or the retailers and on their transaction costs associated with obtaining the product. Moreover, we recall that the demands at the demand markets are aleatory for the higher levels of the network.

We assume that demand at the final market level (i.e., from end customers) is known with certainty, in contrast to the intermediate levels (e.g., retailers and manufacturers), where demand is modeled as stochastic. This modeling choice is grounded in both empirical evidence and structural considerations.

First, in many mature industrial contexts, such as manufacturing, pharmaceuticals, and the automotive sector, historical demand data from end customers are abundant, granular, and highly reliable, owing to the widespread use of advanced monitoring systems and predictive analytics. Second, in several B2B settings, demand is determined through binding supply contracts that stipulate fixed volumes or minimum delivery thresholds. Such agreements significantly reduce uncertainty and justify the adoption of a deterministic representation of demand at the final tier. This approach improves computational efficiency while preserving the model's structural validity.

Let $\hat{c}_{moi}$ and $\hat{c}_{loi}$ be the unit transaction costs associated with consumers at demand market o that buy finished product from manufacturer m and retailer l, respectively, with the shipment mode i, $o = 1,\dots,O,\ m = 1,\dots,M,\ l = 1,\dots,L,\ i = 1,\dots,I$. To express the fact that these costs depend both on the amount of product obtained from all manufacturers and all retailers of the network, we assume that:

$$\hat{c}^t_{moi} \equiv \hat{c}^t_{moi}(g^1, q), \quad \forall m = 1,\dots,M,\ o = 1,\dots,O,\ i = 1,\dots,I,$$

$$\hat{c}^T_{loi} \equiv \hat{c}^T_{loi}(g^1, q) \quad \forall l = 1,\dots,L,\ o = 1,\dots,O,\ i = 1,\dots,I.$$

Let ρ^4_{om} and ρ^4_{ol} be the price of the product that consumers at the demand market o are willing to pay to buy finished product from manufacturer m and retailer l, respectively, and we group these quantities into the vectors $\bar{\rho}^4 = (\rho^4_{mo})_{m=1,\dots,M,o=1,\dots,O} \in \mathbb{R}^{MO}_+$ and $\tilde{\rho}^4 = (\rho^4_{lo})_{l=1,\dots,L,o=1,\dots,O} \in \mathbb{R}^{LO}_+$
To better describe the competition among demand markets, denoting by d_o the request for the product at the demand market o, we assume that such a demand depends on the whole vector $(\bar{\rho}^4, \tilde{\rho}^4) \in \mathbb{R}^{O(M+L)}_+$. As a result, the demand in each market depends not only on its own price, but also on the prices set in the other demand markets. Hence,

$$d_o = \sum_{m=1}^{M} d_{mo}(\bar{\rho}^4_{mo}) + \sum_{l=1}^{L} d_{lo}(\tilde{\rho}^4_{lo}) = d_o(\bar{\rho}^4, \tilde{\rho}^4), \quad \forall o = 1,\dots,O.$$

Therefore, when purchasing the product from a manufacturer, consumers at the demand markets incur both the manufacturer's selling price and the associated transportation costs.

Hence, we have the following Definition for the equilibrium conditions.

Definition 2.7 *A vector $(g^{1*}_o, q^*_o, \bar{\rho}^{4*}_o, \tilde{\rho}^{4*}_o) \in \mathbb{R}^{MI+LI+M+L}_+$ is an equilibrium for consumers at the demand market o if the following conditions hold:*

$$\bar{p}^{2*}_{moi} + \hat{c}^t_{moi}(g^{1*}, q^*) \begin{cases} = \bar{\rho}^{4*}_{mo} & \text{if } g^*_{moi} > 0 \\ \geq \bar{\rho}^{4*}_{mo} & \text{if } g^*_{moi} = 0 \end{cases}, \quad \forall m = 1,\dots,M,\ o = 1,\dots,O\, i = 1,\dots,I, \tag{66}$$

$$\tilde{p}^{3*}_{loi} + \hat{c}^T_{loi}(g^{1*}, q^*) \begin{cases} = \tilde{\rho}^{4*}_{lo} & \text{if } q^*_{loi} > 0 \\ \geq \tilde{\rho}^{4*}_{lo} & \text{if } q^*_{loi} = 0 \end{cases} \quad \forall l = 1,\dots,L,\ o = 1,\dots,O\, i = 1,\dots,I, \tag{67}$$

and

$$d_{mo}(\bar{\rho}^{4*}_{mo}) \begin{cases} = \sum_{i=1}^{I} g^*_{moi} & \text{if } \bar{\rho}^{4*}_{mo} > 0 \\ \leq \sum_{i=1}^{I} g^*_{moi} & \text{if } \bar{\rho}^{4*}_{mo} = 0 \end{cases}, \quad \forall m = 1,\dots,M,\ o = 1,\dots,O, \tag{68}$$

$$d_{lo}(\tilde{\rho}^{4*}_{lo}) \begin{cases} = \sum_{i=1}^{I} q^*_{loi} & \text{if } \tilde{\rho}^{4*}_{lo} > 0 \\ \leq \sum_{i=1}^{I} q^*_{loi} & \text{if } \tilde{\rho}^{4*}_{lo} = 0 \end{cases}, \quad \forall l = 1,\dots,L,\ o = 1,\dots,O. \tag{69}$$

Condition (66) states that consumers at the demand market o buy from a manufacturer m if the sum of the manufacturer's selling price and the transaction costs (incurred by the consumers) is equal to the price that the demand markets are willing to pay. Otherwise, the consumers at the demand market o decide not to buy from the specific manufacturer.

Condition (67) is analogous to the previous condition for each retailer (and each shipping method) and the demand market o.

Condition (68) expresses that, if the price imposed by the consumers at demand market o for purchase from manufacturer m is positive (and, hence, a purchase has been made), the quantity purchased by the consumers at the demand market o from manufacturer m is equal to the demand.

Finally, condition (69) is analogue to condition (68) but for the purchase from retailer l.

The equilibrium conditions (66)-(69) for customers at all the demand markets simultaneously are generalizations of classical spatial price equilibrium conditions due to Samuelson [58] and Takayama and Judge [64] amd are characterized by the following Theorem (see [12], [27] and [68]).

Theorem 2.1 *A vector* $\hat{\psi}^* \equiv (g_o^{1*}, q_o^{1*}, \bar{\rho}_o^{4*}, \tilde{\rho}_o^{4*}) \in \hat{\mathbb{K}}$ *satisfies the equilibrium conditions* (66)-(69) *if and only if is a solution to the variational inequality: find* $\hat{\psi}^* \in \hat{\mathbb{K}}$ *such that:*

$$
\begin{aligned}
&\sum_{m=1}^{M}\sum_{o=1}^{O}\sum_{i=1}^{I}\left[\bar{p}_{moi}^{2*} + \hat{c}_{moi}^{t}(g^{1*}, q^*) - \bar{\rho}_{mo}^{4*}\right] \times (g_{moi} - g_{moi}^*)\\
&+\sum_{l=1}^{L}\sum_{o=1}^{O}\sum_{i=1}^{I}\left[\tilde{p}_{loi}^{3*} + \hat{c}_{loi}^{T}(g^{1*}, q^*) - \tilde{\rho}_{lo}^{4*}\right] \times (q_{loi} - q_{loi}^*)\\
&+\sum_{o=1}^{O}\left[\sum_{m=1}^{M}\left[\sum_{i=1}^{I} g_{moi}^* - d_{mo}(\bar{\rho}_{mo}^{4*})\right]\right] \times (\bar{\rho}_{mo}^{4} - \bar{\rho}_{mo}^{4*})\\
&+\sum_{o=1}^{O}\left[\sum_{l=1}^{L}\left[\sum_{i=1}^{I} q_{loi}^* - d_{lo}(\tilde{\rho}_{lo}^{4*})\right]\right] \times (\tilde{\rho}_{lo}^{4} - \tilde{\rho}_{lo}^{4*}) \geq 0, \quad \forall \hat{\psi} \in \hat{\mathbb{K}},
\end{aligned}
\tag{70}
$$

where $\hat{\mathbb{K}} := \mathbb{R}_+^{O(MI+LI+M+L)}$.

Variational inequality (70) represents the equilibrium for all demand markets simultaneously. The solution gives the equilibrium amount of product that customers at each demand market $o = 1, \ldots, O$ buy from manufacturers (g_o^{1*}) and retailers through each shipping method ((q_o^{1*})). Moreover, the solution gives the equilibrium prices that each demand market ($\bar{\rho}_o^{4*}, \tilde{\rho}_o^{4*}$).

Variational inequality (70) can be put into standard form, that is:
determine $\hat{X}^* \in \hat{\mathbb{K}}$ such that:

$$
\langle \hat{F}(\hat{X}^*), \hat{X} - \hat{X}^* \rangle \geq 0, \quad \forall \hat{X} \in \hat{\mathcal{K}}, \tag{71}
$$

where $\hat{\mathcal{K}}$ is a closed and convex set.

To this purpose, we define $\hat{X} \equiv \hat{\psi}$, $\hat{F} = (\hat{F}^1(\hat{X}), \hat{F}^2(\hat{X}), \hat{F}^3(\hat{X}), \hat{F}^4(\hat{X}))$ and $\hat{\mathcal{K}} \equiv \hat{\mathbb{K}}$, where:

$$
\hat{F}_{moi}^1(\hat{X}) = \left[\bar{p}_{moi}^2 + \hat{c}_{moi}^t(g^1, q) - \bar{\rho}_{mo}^4\right], \quad \forall m, o, i;
$$

$$
\hat{F}_{loi}^2(\hat{X}) = \left[\tilde{p}_{loi}^3 + \hat{c}_{loi}^T(g^1, q) - \tilde{\rho}_{lo}^4\right], \quad \forall l, o, i;
$$

$$
\hat{F}_{mo}^3(\hat{X}) = \left[\sum_{i=1}^{I} g_{moi} - d_{mo}(\bar{\rho}_{mo}^4)\right], \quad \forall m, o;
$$

$$
\hat{F}_{lo}^4(\hat{X}) = \left[\sum_{i=1}^{I} q_{loi} - d_{lo}(\tilde{\rho}_{lo}^4)\right], \quad \forall l, o.
$$

Under the imposed assumptions on the cost and demand functions, we can deduce that the function $\hat{F}(\hat{X})$ defining variational inequality (71) is continuous. Moreover, assuming a coercivity condition on vector function $\hat{F}$, the existence of a solution to variational inequality (71) is guaranteed by the classical theory of variational inequalities (see [23]).

3 Comprehensive Variational Formulation

In this section, we present a unified variational formulation that encompasses the entire FVPN, consisting of the raw material suppliers, the manufacturers, the retailers, and the demand markets.

To ensure consistency across the network tiers, the quantity of raw materials supplied by supplier n (via shipping method i) to manufacturer m must coincide with the quantity received by manufacturer m from supplier n. Therefore, the optimal solution (x^*_{nmi}, y^*_{nmi}) to variational inequality (13) must be the same of variational inequality (42). Likewise, the quantity of product shipped by manufacturer m (via shipping method i) to retailer l must equal the quantity received by retailer l from manufacturer m. Consequently, the optimal solution g^*_{mli} that solves variational inequality (42) must also satisfy variational inequality (63).

A similar correspondence holds for the quantities exchanged between manufacturers and demand markets: the optimal values g^*_{moi} appearing in (13) and (70) must be equal.

Similar considerations can be deduced for the quantities transacted between retailers and demand markets layers, that is q^*_{loi}, appearing in variational inequalities (63) and (70).

Therefore, a global solution for the full FPVN can be obtained by solving a single variational inequality resulting from the aggregation of variational inequalities (13), (42), (63) and (70). This comprehensive formulation ensures coherence across all tiers and formally captures the inter-tier agreements. The following definition provides a formal statement of this unified approach.

Definition 3.1 *An equilibrium state of the presented FVPN is reached if the flows between the tiers of the network coincide and the product shipments and prices satisfy the sum of the conditions* (13), (42), (63) *and* (70).

The following result represents a variational inequality formulation of the governing equilibrium conditions according to Definition 3.1 (see [11] and [41] for the proof). It provides a unifying characterization of the equilibrium of the entire FPVN and guarantees that the individual optimality conditions of each agent, together with the feasibility and coupling constraints (including environmental ones), are satisfied simultaneously.

Theorem 3.1 *A vector* $(x^*, y^*, x^{1*}, y^{1*}, g^{1*}, g^{2*}, q^*, \bar{\rho}^{4*}, \tilde{\rho}^{4*}) \in \mathcal{K}$ *is an equilibrium state of the FVPN according to Definition* (3.1) *if and only if such a vector is a solution to the variational inequality:*
Find $(x^*, y^*, x^{1*}, y^{1*}, g^{1*}, g^{2*}, q^*, \bar{\rho}^{4*}, \tilde{\rho}^{4*}) \in \mathcal{K}$ *such that:*

$$\sum_{n=1}^{N}\sum_{r=1}^{R}\left[\frac{\partial c^1_{nr}(x^*_{nr})}{\partial x_{nr}} + \frac{\partial c^d_{nr}(x^*_{nr}, y^*_{nr}, x^{1*}_{nr}, y^{1*}_{nr})}{\partial x_{nr}} + \alpha_n\left(\frac{\partial e^1_{nr}(x^*_{nr})}{\partial x_{nr}} + \frac{\partial e^d_{nr}(x^*_{nr}, y^*_{nr}, x^{1*}_{nr}, y^{1*}_{nr})}{\partial x_{nr}}\right)\right] \times (x_{nr} - x^*_{nr})$$
$$+\sum_{n=1}^{N}\sum_{r=1}^{R}\left[\frac{\partial c^2_{nr}(y^*_{nr})}{\partial y_{nr}} + \frac{\partial c_{nr}(y^*_{nr})}{\partial y_{nr}} + \frac{\partial c^d_{nr}(x^*_{nr}, y^*_{nr}, x^{1*}_{nr}, y^{1*}_{nr})}{\partial y_{nr}}\right.$$
$$\left. +\alpha_n\left(\frac{\partial e^2_{nr}(y^*_{nr})}{\partial y_{nr}} + \frac{\partial e^d_{nr}(x^*_{nr}, y^*_{nr}, x^{1*}_{nr}, y^{1*}_{nr})}{\partial y_{n}}\right)\right] \times (y_{nr} - y^*_{nr})$$
$$+\sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I}\left[\frac{\partial c^{1t}_{nrmi}(x^*_{nrmi})}{\partial x_{nrmi}} + \frac{\partial c^d_{nr}(x^*_{nr}, y^*_{nr}, x^{1*}_{nr}, y^{1*}_{nr})}{\partial x_{nrmi}} + \alpha_n\left(\frac{\partial e^1_{nrmi}(x^*_{nrmi})}{\partial x_{nrmi}} + \frac{\partial e^d_{nr}(x^*_{nr}, y^*_{nr}, x^{1*}_{nr}, y^{1*}_{nr})}{\partial x_{nrmi}}\right)\right.$$
$$\left. + \frac{\partial \bar{c}^{1s}_m(x^*_m, y^*_m)}{\partial x_{nrmi}} + \frac{\partial \bar{c}^t_{nrmi}(x^*_{nrmi}, y^*_{nrmi})}{\partial x_{nrmi}}\right] \times (x_{nrmi} - x^*_{nrmi})$$
$$+\sum_{n=1}^{N}\sum_{r=1}^{R}\sum_{m=1}^{M}\sum_{i=1}^{I}\left[\frac{\partial c^{2t}_{nrmi}(y^*_{nrmi})}{\partial y_{nrmi}} + \frac{\partial c^d_{nr}(x^*_{nr}, y^*_{nr}, x^{1*}_{nr}, y^{1*}_{nr})}{\partial y_{nrmi}} + \alpha_n\left(\frac{\partial e^2_{nrmi}(y^*_{nrmi})}{\partial y_{nrmi}} + \frac{\partial e^d_{nr}(x^*_{nr}, y^*_{nr}, x^{1*}_{nr}, y^{1*}_{nr})}{\partial y_{nmi}}\right)\right.$$
$$\left. +\frac{\partial \bar{c}^{1s}_m(x^*_m, y^*_m)}{\partial y_{nrmi}} + \frac{\partial \bar{c}^t_{nrmi}(x^*_{nrmi}, y^*_{nrmi})}{\partial y_{nrmi}}\right] \times (y_{nrmi} - y^*_{nrmi})$$
$$+\sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{i=1}^{I}\left[\frac{\partial \bar{c}^{2s}_m(g^{1*}_m, g^{2*}_m)}{\partial g_{mli}} + \frac{\partial \bar{c}^t_{mli}(g^*_{mli})}{\partial g_{mli}} + \bar{\alpha}_m\left(\frac{\partial \bar{e}_m(g^{1*}_m, g^{2*}_m)}{\partial g_{mli}} + \frac{\partial \bar{e}^1_{mli}(g^*_{mli})}{\partial g_{mli}}\right) + \delta^-_m\frac{\partial \mathbb{E}[\Delta^-_m]}{\partial g_{mli}} + \delta^+_m\frac{\partial \mathbb{E}[\Delta^+_m]}{\partial g_{mli}}\right.$$
$$\left. +\frac{\partial \tilde{c}^T_{mli}(g^*_{mli})}{\partial g_{mli}} + \frac{\partial \tilde{c}^s_l(g^{1*}_l, q^*_l)}{\partial g_{mli}} + \tilde{\alpha}_l\frac{\partial \tilde{e}_l(g^{1*}_l, q^*_l)}{\partial g_{mli}} + \tilde{\delta}^-_l\frac{\partial \mathbb{E}[\Delta^-_l]}{\partial g_{mli}} + \tilde{\delta}^+_l\frac{\partial \mathbb{E}[\Delta^+_l]}{\partial g_{mli}}\right] \times (g_{mli} - g^*_{mli})$$

$$
\begin{aligned}
&+\sum_{m=1}^{M}\sum_{o=1}^{O}\sum_{i=1}^{I}\Bigg[\frac{\partial \bar{c}_m^{2s}(g_m^{1*},g_m^{2*})}{\partial g_{moi}}+\frac{\partial \bar{c}_{moi}^{T}(g_{moi}^{*})}{\partial g_{moi}}+\bar{\alpha}_m\left(\frac{\partial \bar{e}_m(g_m^{1*},g_m^{2*})}{\partial g_{moi}}+\frac{\partial \bar{e}_{moi}^{2}(g_{moi}^{*})}{\partial g_{moi}}\right)\\
&\qquad+\delta_m^{-}\frac{\partial \mathbb{E}[\Delta_m^{-}]}{\partial g_{moi}}+\delta_m^{+}\frac{\partial \mathbb{E}[\Delta_m^{+}]}{\partial g_{moi}}+\hat{c}_{moi}^{t}(g^{1*},q^{*})-\bar{\rho}_{mo}^{4*}\Bigg]\times(g_{moi}-g_{moi}^{*})\\
&+\sum_{l=1}^{L}\sum_{o=1}^{O}\sum_{i=1}^{I}\Bigg[\frac{\partial \tilde{c}_{loi}^{t}(q_{loi}^{*})}{\partial q_{loi}}+\frac{\partial \tilde{c}_l^{s}(g_l^{1*},q_l^{*})}{\partial q_{loi}}+\tilde{\alpha}_l\left(\frac{\partial \tilde{e}_{loi}(q_{loi}^{*})}{\partial q_{loi}}+\frac{\partial \tilde{e}_l(g_l^{1*},q_l^{*})}{\partial q_{loi}}\right)\\
&\qquad+\tilde{\delta}_l^{-}\frac{\partial \mathbb{E}[\Delta_l^{-}]}{\partial q_{loi}}+\tilde{\delta}_l^{+}\frac{\partial \mathbb{E}[\Delta_l^{+}]}{\partial q_{loi}}+\hat{c}_{loi}^{T}(g^{1*},q^{*})-\tilde{\rho}_{lo}^{4*}\Bigg]\times(q_{loi}-q_{loi}^{*})\\
&+\sum_{o=1}^{O}\left[\sum_{m=1}^{M}\left[\sum_{i=1}^{I}g_{moi}^{*}-d_{mo}(\bar{\rho}_{mo}^{4*})\right]\right]\times(\bar{\rho}_{mo}^{4}-\bar{\rho}_{mo}^{4*})\\
&+\sum_{o=1}^{O}\left[\sum_{l=1}^{L}\left[\sum_{i=1}^{I}q_{loi}^{*}-d_{lo}(\tilde{\rho}_{lo}^{4*})\right]\right]\times(\tilde{\rho}_{lo}^{4}-\tilde{\rho}_{lo}^{4*})\geq 0,\quad \forall(x,y,g^1,g^2,q,\bar{\rho}^4,\tilde{\rho}^4)\in\mathcal{K},
\end{aligned}
\tag{72}
$$

where the feasible set

$$
\mathcal{K}:=\mathbb{K}\cap\bar{\mathbb{K}}\cap\tilde{\mathbb{K}}\cap\hat{\mathbb{K}}\subseteq\mathbb{R}_{+}^{2NR+2NRI+MLI+MOI+LOI+MO+LO}. \tag{73}
$$

Variational inequality (72) can be put in standard form, as follows:
find $X^* \in \mathcal{K}$ such that:

$$
\langle \mathcal{F}(\mathcal{X}),\mathcal{X}-\mathcal{X}^*\rangle \geq 0,\quad \forall \mathcal{X}\in\mathcal{K}, \tag{74}
$$

were:

- $\mathcal{F}(\mathcal{X})=(\mathcal{F}^a(\mathcal{X}))_{a=1,\dots,9}$ is a vector function with

  $$\mathcal{F}_{nr}^1(\mathcal{X})=F_{nr}^1(\mathcal{X}),\ \mathcal{F}_{nr}^2(X)=F_{nr}^2(\mathcal{X}),\quad \forall n,r$$

  representing the (n,r)-th component of $\mathcal{F}^1$ and $\mathcal{F}^2$, respectively;

  $$\mathcal{F}_{nrmi}^3(\mathcal{X})=F_{nrmi}^3(\mathcal{X})+\bar{F}_{nrmi}^3(\mathcal{X}),\quad \forall n,r,m,i,$$

  representing the $(n,,r,m,i)$-th component of $\mathcal{F}^3$;

  $$\mathcal{F}_{nrmi}^4(\mathcal{X})=F_{nrmi}^4(\mathcal{X})+\bar{F}_{nrmi}^4(\mathcal{X}),\quad \forall n,r,m,i,$$

  representing the (n,r,m,i)-th component of $\mathcal{F}^4$;

  $$\mathcal{F}_{mli}^5(\mathcal{X})=\bar{F}_{mli}^1(\mathcal{X})+\tilde{F}_{mli}^1(\mathcal{X}),\quad \forall m,l,i,$$

  representing the (m,l,i)-th component of $\mathcal{F}^5$;

  $$\mathcal{F}_{moi}^6(\mathcal{X})=\bar{F}_{moi}^2(\mathcal{X})+\hat{F}_{moi}^1(\mathcal{X}),\quad \forall m,o,i,$$

  representing the (m,o,i)-th component of $\mathcal{F}^6$;

  $$\mathcal{F}_{loi}^7(\mathcal{X})=\tilde{F}_{loi}^2(\mathcal{X})+\hat{F}_{loi}^2(\mathcal{X}),\quad \forall l,o,i,$$

  representing the (l,o,i)-th component of $\mathcal{F}^7$;

  $$\mathcal{F}_{mo}^8(\mathcal{X})=\hat{F}_{mo}^3(\mathcal{X}),\quad \forall m,o,$$

  representing the (m,o)-th component of $\mathcal{F}^8$;

  $$\mathcal{F}_{lo}^9(\mathcal{X})=\hat{F}_{lo}^4(\mathcal{X}),\quad \forall l,o,$$

  representing the (l,o)-th component of $\mathcal{F}^9$;

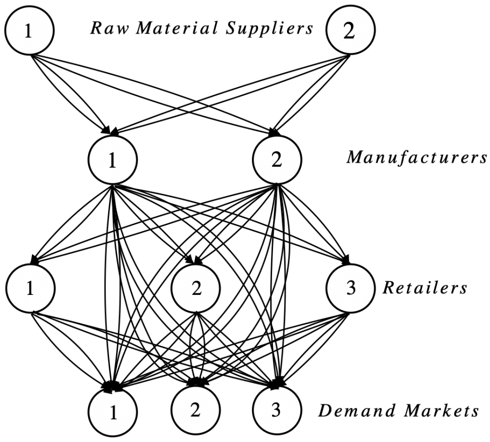


Figure 2: Flexible Production Value Network for the Numerical Simulations

- $\mathcal{X}$ is the $2NR + 2NRI + MLI + MOI + LOI + L + MO + LO$-dimensional vector given by $\mathcal{X} \equiv (x, y, x^1, x^2, g^1, g^2, q, \bar{\rho}^4, \tilde{\rho}^4)$.

We now provide the following theorem, representing an existence and uniqueness result for the solution to variational inequality (72), or variational inequality (74), for whose proof we refer to [23] or [27].

Theorem 3.2 (Existence and Uniqueness) *Under the imposed assumptions on all cost and emission functions, variational inequality* (72)*, or equivalently* (74)*, admits at least one solution. Moreover, if the operator of the variational inequality* (74) *is strictly monotone, that is:*

$$\langle \mathcal{F}(\mathcal{X}^1) - \mathcal{F}(\mathcal{X}^2), \mathcal{X}^1 - \mathcal{X}^2 \rangle > 0, \quad \forall \mathcal{X}^1, \mathcal{X}^2 \in \mathcal{K}, \quad \mathcal{X}^1 \neq \mathcal{X}^2,$$

then the solution $\mathcal{X}^* \in \mathcal{K}$ to variational inequality (74) *is unique.*

4 Numerical Simulations

In this section, we present two different illustrative simulations designed to highlight key features of the proposed FVPN model and evaluate its practical effectiveness.

We report the numerical results for a flexible production value network consisting of two raw material suppliers, two manufacturers, three retailers, and three demand markets, as shown in Figure 2. We also take into account two different shipping methods to transport goods between the levels of the network. In particular, we assume that one of the shipping methods is high-emission (that is, the conventional transportation method), denoted by $i = 1$, while the other one consists in electrical and/or low-emission means of transport, denoted by $i = 2$, and, hence, is environmentally friendly.

The numerical examples are based on synthetic but realistic data designed to illustrate the behavior of the proposed FPVN model under heterogeneous environmental and operational conditions. All cost and emission coefficients have been generated within plausible industrial ranges consistent with previous works on supply chain equilibrium models ([28], [45]). In particular, quadratic cost coefficients reflect increasing marginal costs of production and transportation, whereas linear emission terms capture proportional environmental impacts across transportation modes and production tiers.

The capacity limits, penalty parameters for over- and under-production, and environmental thresholds were calibrated to ensure that equilibrium solutions are feasible and non-trivial. Demand functions were specified as decreasing linear functions of price, with stochastic perturbations modeled through a normal distribution. All data are reported in Section 4.1, and the corresponding parameters are dimensionless, serving purely illustrative purposes. This setting enables full reproducibility of the results while maintaining generality of the conclusions.

In the first numerical simulation, we consider a single type of raw material, i.e $R = 1$. In the second, we consider two different types of raw material.

The results of all the simulations are computed by solving the variational inequality given in the previ-

x^*_n	x^*_1	x^*_2																
Optimal Values	63.14	31.10																
y^*_n	y^*_1	y^*_2																
Optimal Values	18.68	11.30																
x^*_{nmi}	x^*_{111}	x^*_{112}	x^*_{121}	x^*_{122}	x^*_{211}	x^*_{212}	x^*_{221}	x^*_{222}										
Optimal Values	7.04	13.61	7.45	6.86	8.46	12.70	1.27	2.09										
y^*_{nmi}	y^*_{111}	y^*_{112}	y^*_{121}	y^*_{122}	y^*_{211}	y^*_{212}	y^*_{221}	y^*_{222}										
Optimal Values	7.04	13.61	7.46	18.72	5.33	9.44	1.03	2.05										
g^*_{mli}	g^*_{111}	g^*_{112}	g^*_{121}	g^*_{122}	g^*_{131}	g^*_{132}	g^*_{211}	g^*_{212}	g^*_{221}	g^*_{222}	g^*_{231}	g^*_{232}						
Optimal Values	0.06	19.19	10.35	13.60	7.41	37.55	0	8.92	4.46	21.58	1.21	3.81						
g^*_{moi}	g^*_{111}	g^*_{112}	g^*_{121}	g^*_{122}	g^*_{131}	g^*_{132}	g^*_{211}	g^*_{212}	g^*_{221}	g^*_{222}	g^*_{231}	g^*_{232}						
Optimal Values	1.14	3.67	1.81	2.26	3.29	2.62	0.40	0.75	4.76	4.36	6.08	6.24						
q^*_{loi}	q^*_{111}	q^*_{112}	q^*_{121}	q^*_{122}	q^*_{131}	q^*_{132}	q^*_{211}	q^*_{212}	q^*_{221}	q^*_{222}	q^*_{231}	q^*_{232}	q^*_{311}	q^*_{312}	q^*_{321}	q^*_{322}	q^*_{331}	q^*_{332}
Optimal Values	7.04	10.95	2.32	2.44	1.89	3.51	6.33	8.22	8.55	11.39	5.70	9.78	7.67	8.28	8.16	11.07	6.09	8.70
$\bar{\rho}^*_{mo}$	$\bar{\rho}^*_{11}$	$\bar{\rho}^*_{12}$	$\bar{\rho}^*_{13}$	$\bar{\rho}^*_{21}$	$\bar{\rho}^*_{22}$	$\bar{\rho}^*_{23}$												
Optimal Values	34.30	31.30	32.81	56.49	60.16	61.46												
$\tilde{\rho}^*_{lo}$	$\tilde{\rho}^*_{11}$	$\tilde{\rho}^*_{12}$	$\tilde{\rho}^*_{13}$	$\tilde{\rho}^*_{21}$	$\tilde{\rho}^*_{22}$	$\tilde{\rho}^*_{23}$	$\tilde{\rho}^*_{31}$	$\tilde{\rho}^*_{32}$	$\tilde{\rho}^*_{33}$									
Optimal Values	38.5	31.40	32.00	36.7	38.1	31.23	32.02	36.12	33.90									

Table 1: Algorithmically Computed Optimal Solutions for Example 1

ous section using the Euler method (see [15]). The adopted Euler iterative scheme is a standard approach for solving variational inequality problems. Convergence is ensured under the classical assumptions of continuity and monotonicity of the mapping $\mathcal{F}(x)$ and convexity of the feasible set $\mathcal{K}$, which are satisfied in the proposed FPVN model. We implemented the algorithm in Matlab on a MacBook Air equipped with an Apple M1 chip (8-core CPU), 8 GB of memory, running macOS 15.3.1. The calculations were implemented in MATLAB R2023b.

4.1 Example 1: A Single Type of Raw Material

As discussed before, in this preliminary simulation, we assume that each supplier is capable of providing only a single type of raw material. For the sake of notational simplicity, in displaying equilibrium solutions, parameters, cost and emission functions, we will omit the index corresponding to the specific type of raw material supplied.

This simulation consists of 77 variables and the equilibrium solutions are obtained in 180 s. We assume that the typical cost function c, depending on the variable x, has a general quadratic expression, as follows:

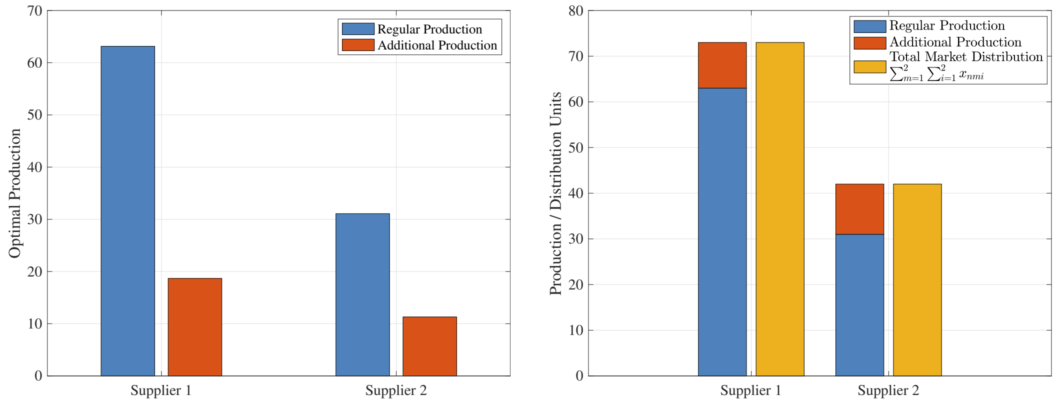
$$c(x) = a_1 x^2 + a_2 x,$$

with $a_1 > 0$ and $a_2 \geq 0$. Observe that in the previous section we assumed that all the cost functions are continuously differentiable and convex and the choice of such an expression and parameters satisfies such assumptions. Moreover, we assume linear emission functions, with a generic expression

$$e(x) = \gamma x, \quad \gamma > 0.$$

The parameters, a_1,a_2 and γ of the cost and emission functions used in the numerical simulations are reported in Tables B.1 and B.2 in Appendix B. The parameters for suppliers, manufacturers, retailers and demand price functions are reported in Appendix B.

The computed solutions are reported in Table 1. These results, derived from the proposed



(a) (b)

Figure 3: (a). Optimal Regular and Additional Production of Suppliers (b). Optimal Production and Total Manufacturer Distribution

multitiered game theory framework, shed light on the operational and strategic behavior of the agents within the flexible production value network. To better interpret the non-cooperative dynamics embedded in the model, we examine the optimal decisions at each tier of the FPVN.

4.1.1 Analysis of Raw Material Suppliers' Optimal Solutions

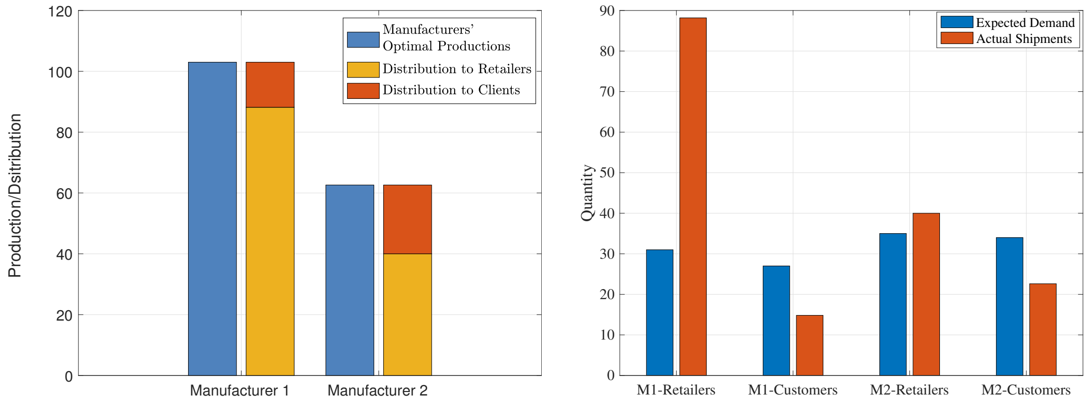
The values of x_n^* and y_n^* in Table 1 and Figure 3(a) represent, respectively, the regular and additional production levels for each of the raw material suppliers. The results show that Supplier 1 has a regular production level of $x_1^* = 63.14$ and an additional production of $y_1^* = 18.68$ for a total output of 81.82 units while Supplier 2 exhibits a regular production of $x_2^* = 31.10$ and an additional production of $y_1^* = 11.30$ totaling 42.40 units.

These results highlight that both producers engage in additional production, indicating that the aggregate demand across markets exceeds the capacity covered by regular production alone. The use of additional production for both producers is a direct consequence of the stochastic nature of demand in the model, which requires a flexible response to uncertain market conditions.

Despite having a significantly higher regular production, Supplier 1 still activates 18.68 units of additional production, suggesting a need to satisfy either unexpectedly high demand in certain markets or shortfalls in downstream distribution. In contrast, Supplier 2, whose regular production is lower, relies even more heavily on additional production in relative terms, approximately 36% of its regular output, compared to about 29% for Supplier 1.

This behavior reflects the model's ability to dynamically allocate additional production based on system-wide needs and local inefficiencies, consistent with the flexible production framework developed in this paper. The use of additional production capacity, despite its higher cost, is essential to ensure supply-demand equilibrium in cases where standard production falls short.

We finally observe that, as shown in Figure 3(b), all the raw material produced by the suppliers, both regular and additional, is fully allocated to the manufacturers. This outcome is primarily driven by economic and environmental incentives embedded in the network. From an economic perspective, suppliers operate under profit-maximizing behavior and face strictly increasing storage costs, which discourage any retention of unsold inventory. Allocating the entire production to manufacturers allows them to avoid unnecessary expenses and to maximize revenue. Furthermore, the presence of stochastic demand in downstream markets induces manufacturers to secure as much raw input as possible to buffer against potential underproduction



(a) (b)

Figure 4: (a). Manufacturers' Optimal Production Compared to the Quantities of Finished Material Shipped to All Retailers and Demand Markets (b). Retailers' Actual Shipments Compared to the Expected Demand

penalties. As a result, suppliers find reliable and immediate demand for their output. Lastly, given the global environmental constraint shared across suppliers, minimizing emissions per unit of useful output becomes a collective priority. Full allocation of available production contributes to this goal by avoiding the environmental cost associated with unutilized or wasted resources. Hence, the complete transfer of material from suppliers to manufacturers emerges as a rational and optimal outcome within the networked production system.

Moreover, raw material suppliers consistently favor low-emission transport modes, reflecting the same sustainability-oriented approach adopted by manufacturers and retailers. The substantial reliance on electric or low-polluting vehicles for raw material deliveries suggests that the internalization of environmental costs and the presence of emission constraints play a decisive role in shaping strategic logistics choices throughout all levels of the network.

4.2 Analysis of Manufacturers' Optimal Solutions

Figure 4(a) illustrates the distribution of manufactured quantities across the two tiers of demand—retailers and end customers—for each manufacturer. A key observation emerging from the plot is that the entirety of each manufacturer's production is allocated across the downstream nodes, with no residual stock or unassigned production. This pattern suggests that the production planning model achieves a full dispatch policy, whereby all available resources are utilized to meet anticipated demand, whether originating from intermediaries (retailers) or final markets.

From a network flow perspective, such a configuration ensures flow conservation at the manufacturing level, indicating a tight coupling between production decisions and logistics. Strategically, this behavior may reflect a cost-optimal allocation structure under the imposed capacity and penalty constraints, where surplus production (i.e., overproduction relative to expected demand) is not stored or wasted, but rather proactively distributed to maximize downstream satisfaction or buffer against demand uncertainty.

This behavior aligns with the fundamental assumptions of the model, particularly the design of the multitiered game theory framework which emphasizes efficient resource deployment in the presence of stochastic demand. It further supports the idea that manufacturers operate under a dispatch-oriented policy, possibly driven by the penalties associated with under- or overproduction, thus pushing the system toward full utilization and minimal idle inventory.

As previously discussed, in order to capture the inherent uncertainty in market and retailer demands, we modeled these quantities as stochastic variables following a normal distribution. Specifically, for each manufacturer m, we considered two aggregate demand sources: d_m^1 and d_m^2, representing the total demand received from retailers, and the direct demand from end customers, respectively. These demands are assumed to follow independent normal distributions, $d_m^k \sim \mathcal{N}(\mu_{km}, \sigma_{km}^2)$, i.e.

$$d_1^1 \sim \mathcal{N}(31, 10),\ d_1^2 \sim \mathcal{N}(35, 10),\ d_2^1 \sim \mathcal{N}(27, 10),\ \ d_1^2 \sim \mathcal{N}(34, 10).$$

Based on the optimal values of the shipped quantities we compute the expected overproduction and underproduction using closed-form expressions derived from properties of the truncated normal distribution.

The numerical analysis and Figure 4(b) reveal that Manufacturer 1 substantially overproduces with respect to retailer demand, leading to an expected surplus of 57.17 units, while simultaneously underproducing relative to final customer demand, resulting in an expected shortfall of 20.17 units. This indicates a production strategy characterized by substantial slack, possibly reflecting a risk-averse behavior or a misalignment between expected and realized demand. For Manufacturer 1, considering a penalty cost associated to a unit of over- and under-production as $\delta_1^- = 0.5$ and $\delta_1^+ = 0.7$ the total penalty for over- and under-production is calculated as $0.7 \times 57.17 + 0.5 \times 20.17$, yielding a penalty of 50.10. This significant penalty underscores the inefficiencies associated with overproducing and underproducing beyond demand, highlighting the cost of excessive or insufficient production that does not align with actual requirements.

Manufacturer 2 exhibits a more constrained production behavior, with an expected underproduction of 13 units with respect to retailer demand and an expected underproduction of 22.62 units with respect to customer demands, as shown in Figure 4(b). The overproduction observed for manufacturer 2, coupled with its underfulfillment of aggregate demand, indicates broader production margins but limited responsiveness to demand fluctuations. Manufacturer 2, considering a penalty cost associated to a unit of over- and under-production as $\delta_2^- = 0.4$ and $\delta_2^+ = 0.5$, incurs a total penalty of $0.5 \times 13 + 0.4 \times 22.62$, amounting to 15.55. While this penalty is smaller in magnitude, it reflects the missed opportunities and potential customer dissatisfaction resulting from failing to meet the required demand. The numerical results provide a comprehensive validation of the proposed model, confirming its capacity to capture strategic interactions among agents in a competitive and environmentally constrained production network. The observed production patterns, particularly for Manufacturer 2, reveal a significant imbalance between actual demand and supply responses. Despite exhibiting a non-negligible level of overproduction, Manufacturer 2 also suffers from notable underproduction, indicating a misalignment between forecasted and realized demand. This dual inefficiency suggests that, while the agent operates with sufficient production flexibility, its responsiveness remains limited, possibly due to conservative planning under uncertainty or suboptimal coordination with downstream actors.

In terms of environmental issues, the numerical solutions reveal a consistent preference for cleaner transportation modes, particularly electric or low-emission vehicles, across multiple FPVN tiers. This behavior emerges despite the decentralized and competitive nature of the network, suggesting that the embedded emission costs and constraints are effective in incentivizing sustainable choices. The observed allocation patterns underscore the potential of market-based mechanisms, such as penalty functions and environmental cost internalization, to steer decentralized agents toward greener logistics strategies without centralized enforcement.

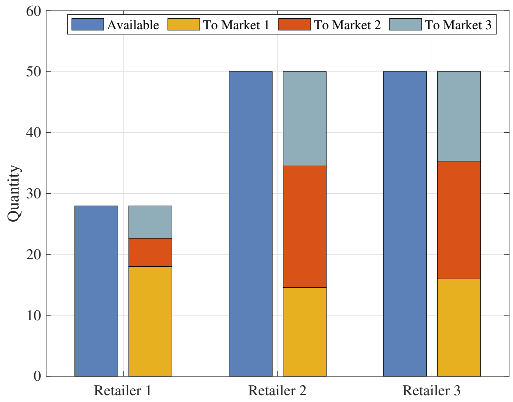


Figure 5: Retailers' Optimal Availability Compared to the Quantities of Finished Material Shipped to All Demand Markets

4.2.1 Analysis of Retailers' Optimal Solutions

Retailer	Expected Understocking	Expected Overstocking
1	8.02	~ 0
2	15.00	~ 0
3	0	17.00

Table 2: Expected Understocking and Overstocking of Retailers

In this section, we extend the stochastic performance analysis to the retailers, adopting the same methodological framework employed for suppliers and manufacturers.

Figure 3 illustrates the disaggregated allocations toward each demand market compared to total availability. A key observation emerging from the plot is that the entirety of each retailer's availability is allocated across the market nodes, with no residual stock or unassigned finished product. This pattern suggests that the shipment and stoking planning model achieves a full dispatch policy, whereby all available resources are utilized to meet anticipated demand.

Each retailer faces an uncertain demand modeled as a normally distributed random variable with parameters estimated from aggregated market-level data. Specifically, demand distributions are defined as $d_l \sim \mathcal{N}(\mu_l, \sigma_l^2)$ where

$$\mu_1 = 36, \sigma_1 = \sqrt{10}, \quad \mu_2 = 65, \sigma_2 = \sqrt{10}, \quad \mu_3 = 33, \sigma_3 = \sqrt{10}$$

denotes the expected demand and the standard deviation for each retailer.

Based on the optimal quantities q^*_{loi} delivered to each market and the absence of safety stocks $s^*_l = 0, \ \forall l$, we compute the expected understocking and overstocking using the standard expressions derived from normal loss functions, summarized in Table 2. The results reveal heterogeneous procurement behaviors across the three retailers. Retailer 1 displays a noticeable expected understocking, while Retailer 2 faces a substantially higher shortfall, indicating a conservative procurement policy that potentially undermines service levels. In contrast, Retailer 3 holds an inventory that significantly exceeds expected demand, resulting in a large expected overstocking. This over-allocation may reflect a risk-averse stance or strategic buffering against demand variability.

These outcomes emphasize the asymmetric risk preferences and operational responses among decentralized decision-makers, and reinforce the necessity of integrating stochastic demand modeling into inventory planning.

Regarding environmental considerations, the computational outcomes consistently indicate a tendency towards the selection of less polluting transport options, notably electric or low-emission vehicles, as in the previous levels of the network. This pattern arises notwithstanding the distributed and competitive characteristics inherent in the network, implying that the integrated emission costs and limitations are influential in promoting environmentally conscious decisions.

4.3 Example 2: Two Types of Raw Material

x^*_{nr}	x^*_{11}	x^*_{21}	x^*_{12}	x^*_{22}														
Optimal Values	54.49	54.76	20.98	40.93														
y^*_{n}	y^*_{11}	y^*_{21}	y^*_{12}	y^*_{22}														
Optimal Values	16.21	19.91	9.63	11.91														
x^*_{nrmi}	x^*_{1111}	x^*_{1112}	x^*_{1121}	x^*_{1122}	x^*_{2111}	x^*_{2112}	x^*_{2121}	x^*_{2122}	x^*_{1211}	x^*_{1212}	x^*_{1221}	x^*_{1222}	x^*_{2211}	x^*_{2212}	x^*_{2221}	x^*_{2222}		
Optimal Values	6.97	13.49	0	21.75	11.34	6.54	9.55	12.91	1.81	4.20	3.42	8.65	5.71	8.81	6.00	8.27		
y^*_{nrmi}	y^*_{1111}	y^*_{1112}	y^*_{1121}	y^*_{1122}	y^*_{2111}	y^*_{2112}	y^*_{2121}	y^*_{2122}	y^*_{1211}	y^*_{1212}	y^*_{1221}	y^*_{1222}	y^*_{2211}	y^*_{2212}	y^*_{2221}	y^*_{2222}		
Optimal Values	6.97	13.49	0	8.00	7.26	4.86	9.30	12.88	1.80	4.20	3.43	3.08	3.48	6.55	5.75	8.24		
g^*_{mli}	g^*_{111}	g^*_{112}	g^*_{121}	g^*_{122}	g^*_{131}	g^*_{132}	g^*_{211}	g^*_{212}	g^*_{221}	g^*_{222}	g^*_{231}	g^*_{232}						
Optimal Values	0	10.59	2.10	5.34	2.81	19.14	1.66	15.98	0.21	12.90	3.36	5.96						
g^*_{moi}	g^*_{111}	g^*_{112}	g^*_{121}	g^*_{122}	g^*_{131}	g^*_{132}	g^*_{211}	g^*_{212}	g^*_{221}	g^*_{222}	g^*_{231}	g^*_{232}						
Optimal Values	1.12	3.62	1.77	2.23	2.58	3.25	0.69	1.03	6.85	7.10	8.90	9.81						
q^*_{loi}	q^*_{111}	q^*_{112}	q^*_{121}	q^*_{122}	q^*_{131}	q^*_{132}	q^*_{211}	q^*_{212}	q^*_{221}	q^*_{222}	q^*_{231}	q^*_{232}	q^*_{311}	q^*_{312}	q^*_{321}	q^*_{322}	q^*_{331}	q^*_{332}
Optimal Values	2.84	2.94	3.85	4.77	3.70	6.45	2.35	2.72	3.36	4.41	2.13	3.62	4.52	5.23	4.96	6.20	3.57	5.50
$\bar{\rho}^*_{mo}$	$\bar{\rho}^*_{11}$	$\bar{\rho}^*_{12}$	$\bar{\rho}^*_{13}$	$\bar{\rho}^*_{21}$	$\bar{\rho}^*_{22}$	$\bar{\rho}^*_{23}$												
Optimal Values	33.73	30.79	32.31	85.27	92.38	93.42												
$\tilde{\rho}^*_{lo}$	$\tilde{\rho}^*_{11}$	$\tilde{\rho}^*_{12}$	$\tilde{\rho}^*_{13}$	$\tilde{\rho}^*_{21}$	$\tilde{\rho}^*_{22}$	$\tilde{\rho}^*_{23}$	$\tilde{\rho}^*_{31}$	$\tilde{\rho}^*_{32}$	$\tilde{\rho}^*_{33}$									
Optimal Values	12.1	27.45	34.32	33.11	27.10	14.32	21.20	33.45	21.20									

Table 3: Algorithmically Computed Optimal Solutions for Example 2

In this second simulation, we extend Example 1 by considering two distinct types of raw materials, each contributing differently to the production of the final product. This setup allows us to capture more realistic production dynamics in which various materials are not perfectly substitutable and may play complementary or asymmetric roles in the manufacturing process.

In this new simulation, we retain the same cost structure and parameter values used in the previous scenario, ensuring consistency in the comparison of outcomes. The additional cost and emission functions introduced to model the two distinct types of raw materials are reported in Table B.3 in Appendix B.

To account for the different contributions of each type of raw material to the production process, we introduce the parameter β_{nrm}, which represents the quantity of raw material of type r required by manufacturer m, supplied by supplier n to manufacture one unit of the final product, $n = 1, 2,\ r = 1, 2,\ m = 1, 2$. This parameter reflects the production-specific bill of materials and allows us to differentiate the roles of each raw material in meeting the final demand. We fix:

$$\beta_{111} = 0.25,\ \beta_{121} = 0.45,\ \beta_{112} = 0.75,\ \beta_{122} = 0.65,$$

$$\beta_{211} = 0.38,\ \beta_{212} = 0.60,\ \beta_{221} = 0.22,\ \beta_{222} = 0.55.$$

For instance, $\beta_{112} = 0.75$ indicates that producer 1, when using raw material of Type 1 supplied by Supplier 1, needs 0.75 units of that material to produce a single unit of output. This implies a relatively high transformation efficiency, since less than one unit of input is needed per unit of output. Conversely, a higher value such as $\beta_{212} = 0.60$ still denotes high efficiency, but slightly lower compared to other combinations (e.g., $\beta_{111} = 0.25$), which indicate even better material utilization in the production process.

The equilibrium solution is reported in Table 3. This simulation consists of 97 variables and the equilibrium solutions are obtained in 273.84 s.

The solution indicates a non-uniform distribution of raw materials across the network, highlighting how the β_{nrm}-parameters shape procurement and transport strategies. To assess how the model allocates raw materials in relation to their transformation efficiency, we examine the equilibrium values of the decision variables x^*_{nrmi} and y^*_{nmri}, which represent the quantities of raw material type r delivered from supplier n to manufacturer m through mode i. These are compared against the corresponding values of the parameter β_{nrm}, which quantifies the amount of raw material required to produce one unit of final product. The results are displayed in Table 4.

Table 4 shows the ranking of supplier–manufacturer–material combinations based on the values of β_{nmr} and the corresponding total allocated quantities $\sum_{i=1}^{2}(x^*_{nrmi} + y^*_{nrmi})$. The results confirm a clear inverse relationship between β_{nmr} and the amount of raw material allocated in the optimal solution. Specifically, the most efficient combinations (those with lower β_{nmr} values) are consistently assigned higher quantities. For example, the combination $(n = 2, m = 2, r = 1)$, with the lowest β_{nmr} value of 0.22, receives the highest allocation (44.64 units), while the least efficient combination $(n = 1, m = 1, r = 2)$, with $\beta_{112} = 0.75$, is allocated only 12 units.

This trend demonstrates that the model effectively captures and leverages transformation efficiency as a key driver in supply chain optimization. The allocation decisions are not random, but clearly influenced by the relative productivity of each supplier–manufacturer pair, in line with the goal of minimizing input requirements and system-wide cost. Moreover, the results show a clear consistency between optimal resource allocation, production efficiency, and environmental interest. Particularly, by comparing the emission coefficients associated with the variables x_{222i}, y_{222i} and x_{221i}, y_{221i}, for $i = 1, 2$, which are identical, it becomes evident that the model tends to favor combinations with lower β_{nrm} values. This implies that, when the environmental impact per transported unit is held constant, production is preferentially allocated to the most efficient suppliers. As a consequence, the overall quantity of raw material required is reduced, leading to a lower level of transport-related emissions.

To contextualize our findings, we compared the outcomes of the proposed FPVN–GNE model with those suggested in related studies. When we disable additional production capacity and assume homogeneous environmental constraints, the equilibrium patterns resemble those described in [20], where flexibility is limited and costs remain stable. Similarly, under closed-loop settings without extra capacity, the results align qualitatively with [18], in which sustainability drives decisions but adaptation margins are reduced. Finally, when emphasizing competition among firms while suppressing regulatory heterogeneity, our results are consistent with [4], where product attributes dominate the equilibrium dynamics. In contrast, when the full FPVN features are activated—heterogeneous environmental constraints and regular versus additional produc-

Supplier n	Material r	Manufacturer m	β_{nrm}	Total Allocated Quantity $\sum_{i=1}^{2}(x^*_{nrmi} + y^*_{nrmi})$	Rank
2	1	2	0.22	44.64	1
1	1	1	0.25	40.92	2
2	1	1	0.38	30	3
1	1	2	0.45	29.75	4
2	2	2	0.55	28.26	5
2	2	1	0.60	24.54	6
1	2	2	0.65	18	7
1	2	1	0.75	12	8

Table 4: Comparison of Transformation Efficiency β_{nrm} and Total Allocated Raw Material Quantities

tion—the simulations reveal distinctive behaviors: firms strategically exploit additional capacity at higher costs, emissions distributions become uneven across tiers, and system-wide trade-offs between profitability and sustainability emerge.

5 Conclusions

In this work, we proposed a comprehensive network-based multitiered game theory framework to analyze and support decision-making in FPVNs under conditions of decentralization, competition, and uncertainty as well as common environmental regulations imposed on tiers of suppliers, manufacturers, and retailers. The proposed model accounts for the strategic behavior of suppliers, manufacturers, and retailers, each acting as a rational agent within a noncooperative environment and subject to coupled economic and environmental constraints.

Through the integration of stochastic demand, nonlinear costs, emission considerations, and equilibrium conditions based on variational inequality theory, the model enables the derivation of equilibrium configurations that reflect realistic interactions in multitier production and distribution systems. The simulation results emphasize several key features: the importance of aligning production and distribution decisions with demand uncertainty, the emergence of endogenous pricing structures, and the potential for inefficiencies such as overproduction and underproduction, all of which are economically interpretable within the variational equilibrium framework.

Furthermore, the analysis highlights the critical role of marginal pricing, penalty mechanisms, and competition across distribution channels in shaping optimal strategies. The coexistence of direct and indirect supply paths, and the decoupling between flow magnitude and price magnitude reveal the expressive capacity of the flexible production value network model in capturing the subtleties of decentralized supply networks.

Overall, this work contributes to the field of sustainable operations and competitive network optimization by offering a flexible and robust modeling approach. However, this study has some limitations that should be acknowledged. The proposed FPVN framework, while capturing essential aspects of flexibility, sustainability, and competition, is built on simplifying assumptions such as convex cost functions, deterministic representations of certain parameters, and fully rational decision-making by agents. These assumptions facilitate the mathematical formulation and tractability of the model but may restrict its direct applicability to real-world decision-making contexts, where bounded rationality, incomplete information, and behavioral considerations are often present. Furthermore, the numerical simulations are illustrative and intended to highlight the main features of the framework, rather than to provide empirical validation on industrial data.

Future research could address these limitations in several directions. A first avenue is the extension of the model to dynamic, multi-period settings, where agents' strategies evolve over time and adapt to changing market conditions. Another promising line of inquiry is the incorporation of stochastic programming elements to better capture uncertainty in demand, costs, and regulatory environments. Integrating behavioral aspects such as bounded rationality, learning-based strategies for demand estimation, or adaptive decision rules could further improve the realism of the framework. Multi-objective formulations may also be explored to explicitly balance cost, service, and environmental performance over extended horizons. An interesting extension would be to incorporate demand-side environmental awareness into the decision processes. In the present formulation, consumers respond primarily to price and availability, with environmental impact embedded in supply-side constraints (e.g., emission bounds at upper tiers of the FPVN). However, in many real-world contexts, consumers—particularly in regulated or socially sensitive markets—consider the carbon footprint or sustainability attributes of products when making purchasing decisions. In addiction, future research could extend the FPVN to explicitly account for emissions generated by warehousing and storage facilities, which represent an additional source of environmental impact linked to energy consumption and operational activities. Finally, empirical validation of the FPVN framework with industrial case studies would provide valuable insights, both for refining the model assumptions and for enhancing its managerial and policy implications.

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References

[1] Allevi, E., Gnudi, A., Konnov, I.V., Oggioni, G., Evaluating the effects of environmental regulations on a closed-loop supply chain network: A variational inequality approach, Annals of Operations Research, 261, 2018, 1-43.

[2] Apple Environmental Progress Report, https://www.apple.com/environment/

[3] Beamon, B. M., Designing the green supply chain, Logistics Information Management, 12 (4), 1999, 332–342.

[4] Besik, D., Nagurney, A., Dutta, P., An integrated multitiered supply chain network model of competing agricultural firms and processing firms, European Journal of Operational Research, 307(1), 2023, 364-381.

[5] Cachon, G., Terwiesch, C., **Matching Supply with Demand: An Introduction to Operations Management**, McGraw-Hill Education, 2023.

[6] Chen, M.C., Pang, S., Su, S.Y., Sustainable global semiconductor supply chain network design considering ESG, Technology in Society, 81, 2025. https://doi.org/10.1016/j.techsoc.2025.102829.

[7] Chopra, S., Meindl, P., **Supply Chain Management: Strategy, Planning, and Operation**, Pearson, 2019.

[8] Colajanni, G., Daniele, P., A convex optimization model for business management, J. Convex Anal., 25(2), 2018, 487–514.

[9] Colajanni, G., Sciacca, D., Paone, L., Optimal energy management of water networks under quality conditions, Intl. Trans. in Op. Res., 2025, https://doi.org/10.1111/itor.70039

[10] Colajanni, G., Daniele, P., Sciacca, D., Reagents and swab tests during the COVID-19 pandemic: an optimized supply chain management with UAVs, Oper. Res. Perspect. (ISSN2214-7160) 9, 2022, 100257, https://doi.org/10.1016/j.orp.2022.100257

[11] Colajanni, G., Daniele, P., Sciacca, D., A variational approach for supply chain networks with environmental interests, EURO Journal on Computational Optimization, Vol. 11, 023, 100075. https://doi.org/10.1016/j.ejco.2023.100075

[12] Dafermos, S., Nagurney, A., Sensitivity analysis for the general spatial economic equilibrium problem, Operations Research, 32, 1984, 1069-1086.

[13] Daniele, P., Sciacca, D., A two-stage variational inequality formulation for a game theory network model for hospitalization in critic scenarios. In: Amorosi, L., Dell'Olmo, P., Lari, I. (eds) **Optimization in Artificial Intelligence and Data Sciences**. AIRO Springer Series, vol 8. Springer, Cham., 2022, pp 13-16. https://doi.org/10.1007/978-3-030-95380-5_2

[14] Debreu, G., A social equilibrium existence theorem, Proc. Natl. Acad. Sci., 38(10), 1952, 886–893.

[15] Dupuis, P., Nagurney, A., Dynamical systems and variational inequalities, Ann. Oper. Res., 44, 1993, 9–42.

[16] Fahimnia, B., Sarkis, J., Davarzani, H., Green supply chain management: A review and bibliometric analysis, International Journal of Production Economics, 162, 2015, 101–114.

[17] Fargetta, G., Maugeri, A., Scrimali, L., A stochastic Nash equilibrium problem for medical supply competition, J. Optim. Theory Appl., 193, 2022, 354–380. https://doi.org/10.1007/s10957-022-02025-y

[18] Fargetta, G., Scrimali, L. R. M., A sustainable dynamic closed-loop supply chain network equilibrium for collectibles markets. Computational Management Science, 20(19), 2023, https:/doi.org/10.1007/s10287-023-00452-4.

[19] Fischer, A., Herrich, M., Schonefeld, K., Generalized Nash equilibrium problems. Recent advances and challenges, Pesquisa Operacional, 34(3), 2014, 521–558.

[20] He, S.-X., Cui, Y.-T., A novel variational inequality approach for modeling the optimal equilibrium in multi-tiered supply chain networks. Supply Chain Analytics, 4, 2023, 100039.

[21] Heckmann, I., Comes, T., Nickel, S., A critical review on supply chain risk–Definition, measure and modelling, Omega, 52, 2015, 119–132.

[22] IKEA Sustainability Report FY22, https://www.ikea.com/global/en/our-business/reports/sustainability-report-fy22-230215/

[23] Kinderlehrer, D., Stampacchia, G., **An Introduction to Variational Inequalities and their Applications**, Academic Press, New York, NY, 1980.

[24] Li, D., Nagurney, A., A general multitiered supply chain network model of quality competition with suppliers, International Journal of Production Economics, 170, 2015, 336-356.

[25] Nagarajan, M., Sosic, G., Game-theoretic analysis of cooperation among supply chain agents: Review and extensions, European Journal of Operational Research, 187 (3), 2008, 719–745.

[26] Nagurney, A., Alternative pollution permit systems for transportation networks based on origin/destination pairs and paths, Transportation Research D, 5, 1999, 37-58.

[27] Nagurney, A., **Network Economics: A Variational Inequality Approach**, Kluwer Academic Publishers, Boston, Massachusetts, 1993.

[28] Nagurney, A., **Supply Chain Network Economics: Dynamics of Prices, Flows, and Profits**, Edward Elgar Publishing, Cheltenham, 2006.

[29] Nagurney, A., Design of sustainable supply chains for sustainable cities, Environ. Plan. B, Plan. Des., 42(1), 2015, 40–57.

[30] Nagurney, A., Supply chain game theory network modeling under labor constraints: Applications to the Covid-19 pandemic, European Journal of Operational Research, 293(3), 2021, 880-891.

[31] Nagurney, A., Daniele, P., Flores, E. A., Caruso, V., A variational equilibrium network framework for humanitarian organizations in disaster relief: Effective product delivery under competition for financial funds. In: Kotsireas, I., Nagurney, A., Pardalos, P. (eds) **Dynamics of Disasters: Algorithmic Approaches and Applications**, 2018, Springer, Cham., pp 109-133. https://doi.org/10.1007/978-3-319-97442-2_6

[32] Nagurney, A., Dhanda, K.K., Marketable pollution permits in oligopolistic markets with transaction costs, Operations Research, 48, 2000, 424-435.

[33] Nagurney, A., Dhanda, K.K., Noncompliant oligopolistic firms and marketable pollution permits: Statics and dynamics, Annals of Operations Research, 95, 2000, 285-312.

[34] Nagurney, A., Dong, J., Zhang, D., A supply chain network equilibrium model, Transportation Research E, 38, 2002, 281-303.

[35] Nagurney, A., Dutta, P., Supply chain network competition among blood service organizations: A Generalized Nash Equilibrium framework, Annals of Operations Research, 275(2), 2019, 551-586.

[36] Nagurney, A., Flores, E.A., Soylu, C., A Generalized Nash Equilibrium network model for post-disaster humanitarian relief, Transportation Research E, 95, 2016, 1-18.

[37] Nagurney, A., Hassani, D., Nivievskyi, O., Martyshev, P., Multicommodity international agricultural trade network equilibrium: Competition for limited production and transportation capacity under disaster scenarios with implications for food security, European Journal of Operational Research, 314(1), 2024, 1127-1142.

[38] Nagurney, A., Ke, K., Financial networks with intermediation, Quantitative Finance, 1(4), 441.

[39] Nagurney, A., Liu, Z., Woolley, T., Optimal endogenous carbon taxes for electric power supply chains with power plants, Mathematical and Computer Modelling, 44, 2006, 899-916.

[40] Nagurney, A., Liu, Z., Woolley, T., Sustainable supply chain networks and transportation, International Journal of Sustainable Transportation, 1, 2007, 29-51.

[41] Nagurney, A., Loo, J., Dong, J., et al., Supply chain networks and electronic commerce: a theoretical perspective, NETNOMICS, 4, 2002, 187–220.

[42] Nagurney, A., Nagurney, L.S., Sustainable supply chain network design: A multicriteria perspective, International Journal of Sustainable Engineering, 3, 2010, 189-197.

[43] Nagurney, A., Qiang, Q., Fragile networks: identifying vulnerabilities and synergies in an uncertain age, International Transactions in Operational Research, 19 (1-2), 2012, 123-160.

[44] Nagurney, A., Ramanujam, P., Dhanda, K.K., A multimodal traffic network equilibrium model with emission pollution permits: Compliance versus noncompliance, Transportation Research D, 3, 1998, 349-374.

[45] Nagurney, A., Saberi, S., Shukla, S., Floden, J., Supply chain network competition in price and quality with multiple manufacturers and freight service providers, Transportation Research E, 77, 2015, 248-267.

[46] Nagurney, A., Salarpour, M., A multicountry, multicommodity stochastic game theory network model of competition for medical supplies inspired by the COVID-19 pandemic. Int. J. Prod. Econ., 235, 2023, 108074.

[47] Nagurney, A., Salarpour, M., Dong, J., Dutta, P., Competition for medical supplies under stochastic demand in the Covid-19 pandemic: A Generalized Nash Equilibrium framework, **Nonlinear Analysis and Global Optimization**, T.M. Rassias, and P.M. Pardalos, Editors, Springer Nature Switzerland AG, 2021, 331-356.

[48] Nagurney, A., Toyasaki, F., Supply chain supernetworks and environmental criteria, Transportation Research D, 8, 2003, 185-213.

[49] Nagurney, A., Toyasaki, F., Reverse supply chain management and electronic waste recycling: A multitiered network equilibrium framework for e-cycling, Transportation Research E, 41, 2005, 1-28.

[50] Nagurney, Q., Yu, M., Sustainable fashion supply chain management under oligopolistic competition and brand differentiation, International Journal of Production Economics, 135, 2012, 532-540.

[51] Nagurney, A., Yu, M., Besik, D., Supply chain network capacity competition with outsourcing: A variational equilibrium framework, J. Global Optim., 69(1), 2017, 231–254.

[52] Nagurney, A., Yu, M., Qiang, Q., Supply chain network design for critical needs with outsourcing. Papers in Regional Science, 90(1), 2011, 123-143.

[53] Pindyck, R. S., Rubinfeld, D. L., **Microeconomics (7th ed.)**, Pearson Prentice Hall, 2009.

[54] Procter & Gamble Ambition 2030, www.pginvestor.com/esg/environmental/environmental-overview/

[55] Qiang, Q., Ke, K., Anderson, T., Dong, J., The closed-loop supply chain network with competition, distribution channel investment, and uncertainties. Omega, 41(2), 2013, 186–194.

[56] Qiang, Q., Nagurney, A., Dong, J., Modeling of supply chain risk under disruptions with performance measurement and robustness analysis. In: Wu, T., Blackhurst, J. (eds) **Managing Supply Chain Risk and Vulnerability: Tools and Methods for Supply Chain Decision Makers**, Springer, London, England, 2009, 91-111.

[57] Saberi, S., Cruz, J.M., Sarkis, J., Nagurney, A., A competitive multiperiod supply chain network model with freight carriers and green technology investment option, European Journal of Operational Research, 266(3), 2018, 934-949.

[58] Samuelson, P.A., Spatial price equilibrium and linear programming, American Economic Review, 42, 1952, 283-303.

[59] Sethi, A.K., Sethi, S.P., Flexibility in manufacturing: A survey, International Journal of Flexible Manufacturing Systems, 2, 1990, 289–328.

[60] Shapiro, J. F., **Modeling the Supply Chain**, Cengage Learning, 2006.

[61] Simchi-Levi, D., Kaminsky, P., Simchi-Levi, E., **Designing and Managing the Supply Chain: Concepts, Strategies, and Case Studies**, McGraw-Hill Education, 2019.

[62] Stecke, K.E., Formulation and solution of nonlinear integer production planning problems for flexible manufacturing systems, Management Science, 29(3), 1983, 273–288.

[63] Tang, C. S., Perspectives in supply chain risk management, International Journal of Production Economics, 103 (2), 2006, 451–488.

[64] Takayama, T., Judge, G.G., **Spatial and Temporal Price and Allocation Models**, North-Holland Publishing Company, Amsterdam, The Netherlands, 1971.

[65] Toyota Sustainability Data Book, https://global.toyota/en/sustainability/report/sdb/

[66] Unilever Annual Report, https://www.unilever.com/investors/annual-report-and-accounts/

[67] Wakolbinger, T., Toyasaki, F., Nowak, T., Nagurney, A., When and for whom would e-waste be a treasure trove? Insights from a network equilibrium model of e-waste flows, International Journal of Production Economics, 154, 2014, 263–273.

[68] Wardrop, J. G., Some theoretical aspects of road traffic research, in: Proceedings of the Institute of Civil Engineers, Part II, 1952, pp 325–378.