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Physical Proof of the Occurrence of the Braess Paradox in Electrical Circuits

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Abstract. - The Braess Paradox is the counterintuitive phenomenon that can occur in a user-optimized network system, such as a transportation network, where adding an additional link to the network increases the cost (travel time) for every user. In electrical circuits, electrons, analogous to drivers in a transportation network, traverse the network so that no electron can unilaterally change its cost (voltage drop) from an origin to a destination. In this paper, we show that the Braess Paradox can occur in electrical circuits consisting of diodes and resistors. We report measurements confirming the occurrence of the Braess Paradox in two different circuits, one with highly nonlinear link cost functions (I-V characteristics). These measurements show that the voltage increases, rather than decreases, when a link is added to the circuit under constant demand (current). This discovery identifies novel circuits in which the voltage and current can be independently adjusted. It also yields insights into the Braess paradox and transportation networks through a new computational mechanism.

Introduction. – The occurrence of counterintuitive behavior in transportation networks was identified in 1968 by Dietrich Braess [1, 2]. In [1], Braess describes a user-optimized transportation network (see also [3–6]) in which, when a link is added, each user traveling in the network becomes worse-off in terms of travel cost/time. This discovery was contrary to the conventional wisdom, which indicated that adding a link, which provides travelers with another route option between their origin/destination pair, would make each user better-off. This counterintuitive phenomenon has become known as the *Braess Paradox* and has been studied by a number of researchers over the past several decades including [7–15]. The Braess Paradox in real systems has been identified in road transportation networks in Stuttgart, Germany, New York City's 42nd St during Earth Day [16], and in Seoul, Korea [17], and in physical network systems such as power and mechanical networks [15, 18–23].

There is, interestingly, a connection between the user-optimized transportation network and electrical circuits. Analogous to vehicles in a user-optimized transportation network who cannot improve their travel times by unilateral action, electrons in an electrical circuit distribute themselves so that the voltage drops on any used path through the circuit are identical. By mapping the Braess cost functions from [1] directly into the voltage drops across an electrical network consisting of ideal resistors and ideal Zener diodes, Cohen and Horowitz [18] predicted that the Braess Paradox could be observed in an electrical network.

The purpose of this work is to show that the Braess Paradox can occur in a macroscopic electrical circuit using real electrical components and can be measured in the laboratory. By exploiting the mathematical analysis for electrical circuits, one can then gain insights into networks that exhibit the Braess Paradox. We formulate the Braess Paradox network as an electrical circuit, present an analysis using Kirchhoff's Laws, and discuss how this analysis yields new insights. We then identify two distinct macroscopic electrical circuit topologies for which we exhibit examples of the occurrence of the Braess Paradox.

We now, for completeness, and easy reference, recall the classical Braess Paradox transportation network example.

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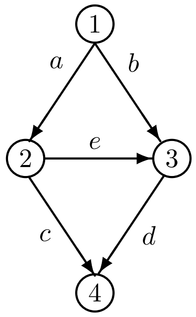

Fig. 1: The Braess Network with Three Paths

Please refer to Figure 1.

The link cost functions are:

$$c_a(f_a) = 10f_a, \;\; c_b(f_b) = f_b + 50,$$

$$c_c(f_c) = f_c + 50, \;\; c_d(f_d) = 10f_d, \;\; c_e(f_e) = f_e + 10,$$

with the flow on link a denoted by f_a, the flow on link b by f_b, and so on, and the origin/destination pair of nodes is (1,4) with a demand of 6 (vehicles per unit time). We define paths p_1, p_2, and p_3, consisting of links, where $p_1 = (a, c)$, $p_2 = (b, d)$, and $p_3 = (a, e, d)$. Clearly, if the flows on the paths are as follows: $x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = 2$ with the user costs on paths being $C_{p_1} = c_a + c_c = C_{p_2} = c_b + c_d = C_{p_3} = c_a + c_e + c_d = 92$, then noone has any incentive to switch, since switching would result in a higher path cost. Note that, without link e, and, hence, path p_3 in the network in Figure 1, the user-optimized path flow pattern would be: $x^*_{p_1} = x^*_{p_2} = 3$ with path costs of: $C_{p_1} = C_{p_2} = 82$. Hence, the addition of link e, which results in the new path p_3 makes all travellers or users of the network worse-off since the cost increases from 83 to 92 with the added path p_3!

While the classical example of the Braess Paradox uses cost functions that are of the form: a fixed term plus a term proportional to the flow, other possible cost functions have been mathematically investigated in transportation networks, including the Bureau of Public Roads cost function, which has a term quartic in the flow [7, 8, 24].

Electrical Circuit Equivalent of the Braess Paradox. – An idealized electrical circuit that exhibits the Braess Paradox behavior is developed by mapping the cost functions on links in Figure 1 into the voltage drops of electrical components as illustrated in Figure 2. The demand on the Braess network is analogous to the current in the electrical circuit in Figure 2, and the cost on a link is analogous to the voltage drop on the corresponding link of the circuit. Let V_i; $i = 1, \ldots, n$, where n is the number of nodes in the electrical circuit, be the voltage at node i referenced to the bottom node of the circuit which we refer to as the reference (or ground) node. A node in an electrical circuit corresponds to a point where two or more components are connected. Let the demand through the electrical network be I and the flow through a link i be I_i. To create the voltage drop analogies for the cost functions,

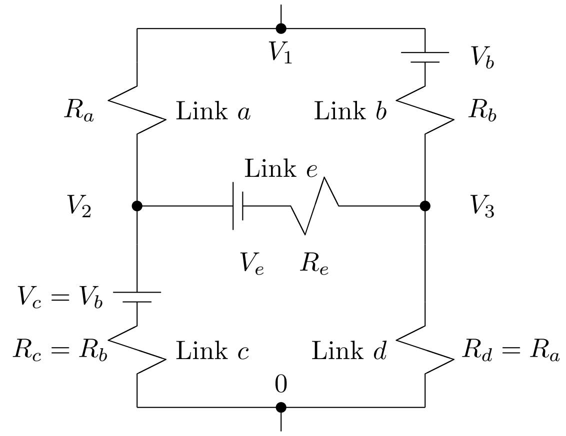

Fig. 2: Electrical Circuit Analogue for the Classical Braess Paradoz

we first note that Ohm's Law states that in a resistor the voltage drop is proportional to the current through that resistor, the proportionality factor is defined as the resistance, R, of that resistor. Thus, a term that has cost proportional to flow is the equivalent of a resistor. Fixed cost terms in a link cost function can be modeled as a constant voltage drop. Symbolically this can be represented as a battery.

Because of the symmetry of the Braess Paradox example, in that: $c_d = c_a$, $c_c = c_b$, we have that $R_d = R_a$, and $V_c = V_b$, $R_c = R_b$.

In the electrical circuit, the voltage, V_1, is the equivalent of the cost for a user (electron) to flow through the circuit. Thus, the Braess Paradox occurs if, by adding link e, the voltage V_1 increases.

The electrical circuit equivalent of the Braess Paradox illustrated in Figure 2 can be analyzed by nodal analysis. Using Kirchhoff's current law at each node i; $i = 1, 2, 3$, in Figure 2, the nodal equations can be written in matrix notation as:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = G^{-1} \begin{bmatrix} I + \frac{V_b}{R_b} \\ -\left(\frac{V_b}{R_b} + \frac{V_e}{R_e}\right) \\ \left(\frac{V_b}{R_b} + \frac{V_e}{R_e}\right) \end{bmatrix}, \tag{1}$$

where G is the conductance matrix

$$G = \begin{bmatrix} \frac{1}{R_a} + \frac{1}{R_b} & -\frac{1}{R_a} & -\frac{1}{R_b}, \\ \frac{1}{R_a} & -\frac{1}{R_a} - \frac{1}{R_b} - \frac{1}{R_e} & \frac{1}{R_e}, \\ \frac{1}{R_a} & \frac{1}{R_e} & -\frac{1}{R_a} - \frac{1}{R_b} - \frac{1}{R_e} \end{bmatrix}.$$

However, for the case when link e is not in the circuit, we observe that, by symmetry, the voltage drop on each of the two paths is the same and that the current splits equally at the top node, resulting in: $I_a = I_b = I_c = I_d = I/2$. In this case, it is straightforward to write the node voltages as

$$\begin{aligned} V_1 = & \; V_b + (R_a + R_b)\tfrac{I}{2} \\ V_2 = & \; V_b + R_b\tfrac{I}{2} \\ V_3 = & \; R_a\tfrac{I}{2}. \end{aligned} \tag{2}$$

In this Kirchhoff's nodal analysis, it is not explicit that the batteries corresponding to V_b and V_e are passive voltage drops and are not generating current. Solutions of the matrix equation might exist where, for example, the flow on link a is greater than the flow I since if the battery supplies current the flow on link b would not be in the direction from node 1 to node 3. While this type of solution is mathematically possible, it does not correspond to a passive electrical circuit example of the Braess Paradox. To verify that the batteries corresponding to V_b and V_e are acting as voltage drops, after computing the solutions for V_1, V_2, and V_3 above, by matrix inversion in equation (3), the flows on links b and c must be calculated. If the two inequalities below are satisfied, then the batteries are acting as voltage drops and the solution corresponds to a passive electrical circuit:

$$\frac{V_1 - V_b - V_3}{R_b} \geq 0 \qquad \text{and} \qquad \frac{V_2 - V_b}{R_b} \geq 0. \tag{3}$$

By substituting in the values from the classical Braess example [1], where $V_b = 50V$, $V_e = 10V$, $R_a = 10\Omega$, $R_b = 1\Omega$, $R_e = 1\Omega$, and $I = 6$, into (1) and (2) and calculating the vector of node voltages, the node voltage V_1 becomes $V_1 = 83V$ without link e in the circuit and $V_1 = 92V$ when link e is in the network. This reproduces the transportation network example in the original Braess article [1, 2].

This Kirchhoff's nodal analysis of the Braess Paradox circuit can also provide additional insights into the conditions on the cost functions of the networks that will exhibit the Braess Paradox. For example, by looking at the right-hand-side of equation (1), one notes that V_e only occurs in the sum $V_b + V_e \frac{R_b}{R_e}$. This indicates that there might be networks that exhibit the Braess Paradox behavior without a fixed cost term in the added link e. This will be shown experimentally later in this paper.

A Circuit that Exhibits the Braess Paradox Using Zener Diodes and Resistors. – It was proposed in [18] that a Wheatstone Bridge circuit consisting of Zener diodes and resistors could exhibit the Braess Paradox. However, [18] noted that their circuit had *unrealistic values in practice, but convenient for illustration*. To construct a circuit with values that could be realized in practice, consider the following. If the conductance matrix, G, is made dimensionless by factoring out R_b^{-1} to become

$$G = \hat{G} R_b^{-1},$$

then (1) becomes

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \hat{G}^{-1} \begin{bmatrix} IR_b + V_b \\ -\left(V_b + V_e \frac{R_b}{R_e}\right) \\ \left(V_b + V_e \frac{R_b}{R_e}\right) \end{bmatrix}. \tag{4}$$

To calculate other possible component values for an electrical circuit that will display the Braess Paradox, we first

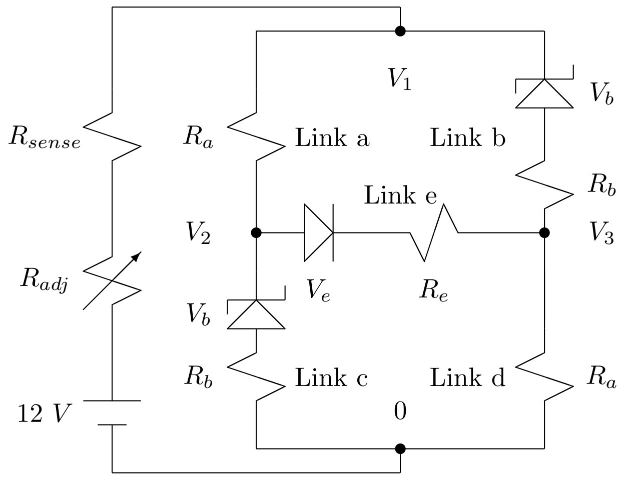

Fig. 3: Electrical Circuit using Zener Diodes for the Braess Paradox Measurements

note that the matrix, $\hat{G}$, depends only on the ratios $\frac{R_b}{R_a}$ and $\frac{R_b}{R_e}$, which in the classical Braess example are .1 and 1, respectively. Because multiplying both sides of a matrix equation by a constant does not change the equation, we can scale V_b, V_e, and the quantity IR_b by the same factor and still have an electrical circuit that exhibits the Braess Paradox behavior. This allows the choice of realistic component values, and current, I. The batteries can then be replaced by Zener diodes, which, to a good approximation, can be modeled as voltage drops.

For example, if we scale both sides of equation (4) by .1, a circuit consisting of

$$V_b = 5V, \qquad V_e = 1V, \qquad I = 6mA,$$

$$R_b = R_e = 100\Omega, \qquad \text{and} \quad R_a = 1000\Omega,$$

should exhibit the Braess Paradox with the voltage $V_1 = 8.3V$ with link e removed and $V_1 = 9.2V$ with link e in the circuit.

To construct a circuit with real components inspired by the classical Braess example, we note that scaled V_b should be near $5V$. The closest standard Zener diode (1N4733A) has Zener voltage 5.1V. Since the ratio of V_b/V_e is 10 in the ideal electrical analogue of the classical Braess example, we select the diode in link e to be a forward biased silicon diode (1N4002) that has a forward voltage drop of approximately .6 V. To choose the value for R_b, we note that the quantity IR_b should equal .6 V. For convenience, we then select $R_b = 100\Omega$ and $I = 6$ mA. Then, since the ratios of R_b/R_a and R_b/R_e in the classical Braess example are .1 and 10, respectively, we select $R_e = R_b = 100$ Ω and $R_a = 10R_b = 1000$ Ω. It is verified computationally through the use of equations (1), (2), and (3), that for these choices of component values, the Braess Paradox is observed in the circuit.

The complete physically constructed circuit, which we use for the first set of Braess Paradox measurements, is shown in the schematic in Figure 3. The current, I, is generated by a fixed voltage 12 V power supply. There are two additional resistors, R_{sense} and R_{adj}, between the fixed 12 V power supply and the Braess circuit. R_{adj} consists of a 680Ω resistor in parallel with a $10K\Omega$ adjustable resistor. The $10K\Omega$ adjustable resistor is adjusted during the measurement to set the current I, the demand, to 6 mA. The 100Ω sense resistor is included so that the current can be determined by measuring the voltage across R_{sense}. For the desired current of 6 mA, the voltage across V_{sense} would be .6 V.

The voltages at nodes V_1, V_2, and V_3, and the voltage across R_{sense} are measured using the 4 analog input channels of a National Instruments USB-6009 Multifunction I/O Data Acquisition system. The USB-6009 system was programmed using Labview running on a PC. From these measurements and the knowledge of the resistor values R_a, R_b, and R_e, the link and path flows are calculated.

For this circuit, five measurements are made for cases corresponding, respectively, to: Case 1: link e absent, Case 2: link e present with $V_e = .62\ V$ and $R_e = 100\ \Omega$ (analogous to the classical Braess example), and three additional measurements with variations on the link e cost functions: Case 3: link e present with only $R_e = 100\Omega$, Case 4: link e present with only $V_e = .62\ V$, and Case 5: link e is a short circuit, i.e., $R_e = 0$. For all cases the cost functions on links $a - d$ are as below:

$$\begin{array}{lll} \text{Cost on link } a & c_a: \ 1000f_a & = 1000I_a \\ \text{Cost on link } b & c_b: \ 5.1 + 100f_b & = 5.1 + 100I_b \\ \text{Cost on link } c & c_c: \ 5.1 + 100f_c & = 5.1 + 100I_c \\ \text{Cost on link } d & c_d: \ 1000f_d & = 1000I_d. \end{array}$$

The experimental results are summarized in Table 1.

Case	V_e	R_e	V_1	Form of Link e Cost Function
1	-	∞	8.13	Link e not in network
2	.62	100	9.21	$V_e + I_eR_e$
3	0	100	9.72	I_eR_e
4	.62	0	8.14	V_e
5	-	0	9.88	Link e is a short circuit

Table 1: Measured Voltage Across an Electrical Circuit using Zener Diodes Exhibiting the Braess Paradox

The results from Table 1 are interpreted as follows. In all cases, when link e is added, the voltage at node 1, V_1, increases, showing that the Braess Paradox occurs in the circuit. In electrical circuits one would normally expect the voltage to drop when a link is added. These multiple examples prove that, in contrast, the opposite can happen. This is the first experimental observation of the Braess Paradox in electrical circuits with physical proof.

The first and second cases correspond to the Braess Paradox transportation example scaled for real components. The first case is the circuit with link e not present. The second case is the circuit with link e added with a voltage drop of $.62\ V + 100I_e$ for flow from node 2 to node 3 (to the right in Figure 3). If the voltage at node 2 was not greater than the voltage at node 3 by .62V, there would be no flow between these two nodes. The cost for the flow through the circuit is 8.13 V in the absence of link e and 9.21V in the presence of link e; thus, confirming the observation of the Braess Paradox in the circuit.

The last three cases of Table 1 correspond to other functional forms of the cost functions for link e, again scaled for real components. The third case in the table corresponds to the case in which link e's cost is only proportional to the flow. From our analysis of equation (4), we note that if the Braess Paradox exists in a circuit for a set of values I, V_b, and V_e, one can choose another set of values, $V_b' = V_b + V_e$, $V_e' = 0$, and $I' = I - (V_e/R_e)$, without changing the right-hand-side of equation (1). Thus, the Braess Paradox does occur in this modified circuit and is measured.

The fourth case of Table 1 is the case where link e is a fixed cost link. Because of the $I - V$ characteristics of a forward biased diode, the actual voltage drop is a sharp exponential, whose limit approaches a fixed value. However, as noted in the previous section, the fixed voltage drop model is only an approximation to the behavior of the real diode and, in fact, the voltage drop on a link will always depend at least slightly upon the current through that link. For this case the circuit only marginally illustrates the Braess Paradox.

Finally, Case 5 in Table 1 corresponds to the case of a zero cost link e. In practice, this was constructed by using a piece of wire for the link. This link may be analyzed as a circuit with a resistor in parallel with the series Zener diode-resistor combination. The measured V_1 in this case is less than either twice the Zener voltage ($10.2V$) or the total current through the resistors, R_a ($12V$), which may be interpreted by assuming non-ideal behavior of the reverse leakage current of a Zener diode.

Electrical Circuit Using Diodes/Resistors. – The extension of the Braess Paradox analysis to other forms of cost functions has been mathematically investigated in [7,8,24]. The driving force for these investigations has been that realistic travel cost functions are based upon the Bureau of Public Roads (BPR) travel cost functions which model the cost on a link as

$$c_a(f_a) = t_a^0 \left(1 + k\left(\frac{f_a}{u_a}\right)^\beta\right), \qquad (5)$$

where t_a^0, k, u_a, and β are positive constants. A full interpretation of the BPR cost functions can be found in [5]. In practice, $k = .15$, $\beta = 4$, and u_a is the practical capacity of link a.

While it is impossible to find a passive electrical component whose $I - V$ characteristics are identical in form to the BPR cost functions, the $I - V$ characteristics of

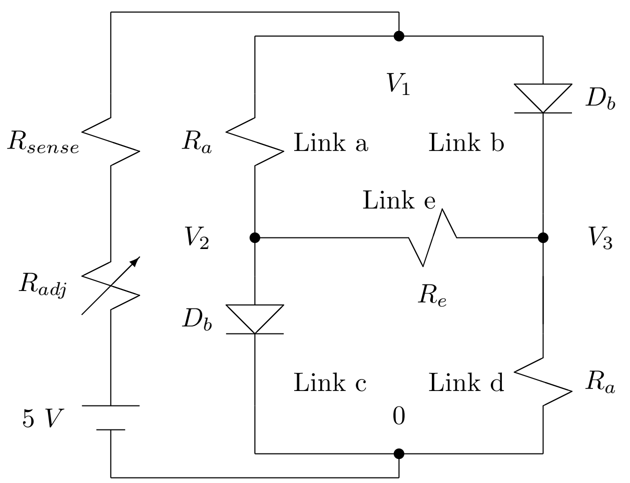

Fig. 4: Diode Resistor Circuit for Braess Paradox Measurement

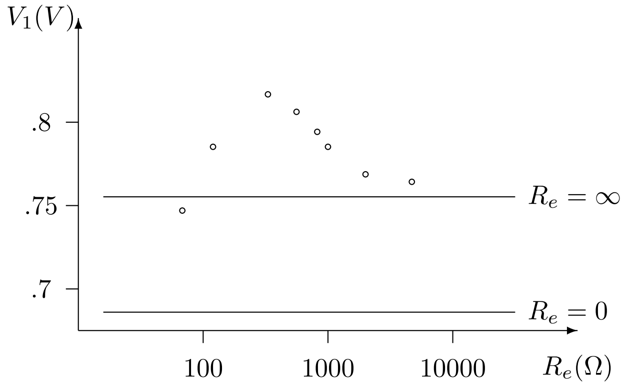

Fig. 5: Measured Node 1 Voltage for a Diode Resistor Circuit

a forward biased diode have an exponential shape. The Shockley diode model predicts that the voltage across any diode, V_D, may be modeled as

$$V_D = V_T \ln\left(\frac{I_D}{I_S}\right), \tag{6}$$

where V_T is the thermal voltage approximately 26 mV at room temperature and I_S is the saturation current approximately 1 pA for a silicon diode.

The first approximation to (6) is the piecewise linear model which models the diode as a voltage source in series with a resistor. This model for the diode voltage drop is identical in form to the voltage drop on links b and c of our initial circuit. Equation (6) can be expanded as a power series in I producing higher order terms similar to those suggested as more complicated transportation cost functions [7,8,24].

Consider the electrical circuit as in Figure 4 with links b and c implemented by forward-biased silicon diodes. This topology was implied in [7,8] but in the context of transportation networks. Unlike the earlier Zener diode electrical circuit example, it is not possible to write a direct matrix equation such as (1) to analyze the circuit. The circuit can, nevertheless, be analyzed by standard electrical circuit simulation software such as SPICE to predict the occurrence of the Braess paradox.

To verify the Braess Paradox in a diode resistor circuit, the circuit in Figure 4 is constructed. Diodes D_b are 1N4148 silicon diodes. The two resistors labeled R_a are 330 Ω. The current (demand), I, is chosen to be 1 mA. The value of the fixed power supply voltage is changed to 5 V and the value R_{sense} is set to 1000Ω. As in the previous experiments, R_{adj} is adjusted to set the current in this case to 1 mA.

The value of R_e is varied from a short circuit, $R_e = 0$, to an open circuit, $R_e = \infty$, i.e., from a zero-cost link to when link e is not in the circuit. The results of the measurements are graphically illustrated in Figure 5. The upper horizontal line in the figure corresponds to the voltage at node 1, V_1, when link e is not included in the system. The lower horizontal line in the figure corresponds to V_1 when link e is a short circuit (wire). The circles are the measured values of V_1 at that value of resistance for link e.

If the Braess Paradox does not occur, the addition of a resistor as link e would lower the voltage at node 1, V_1, which would result in all measured values of V_1 being between the two horizontal lines. As can be seen from Figure 5, for link e resistances greater than 100 Ω, the results for the circuit illustrate that the equivalent resistance of the circuit increases when a resistor is added for link e; thus, confirming Braess Paradox behavior.

Summary. – In this paper, we explore the behavior of electrons flowing through an electrical circuit, which is physically revealed to be governed by the same relationship that governs travelers driving in a road network and seeking their optimal routes of travel from origin nodes to destinations, acting independently. We then prove, through several physically constructed electrical circuits, that the Braess Paradox, originally proposed in user-optimized transportation networks, also can occur in electrical circuits, where the addition of a new link results in an increase in the voltage, rather than a decrease, as is the expectation. We provide examples in which cost functions are both linear as well as highly nonlinear and the same counterintuitive phenomenon is observed.

From an electrical circuit perspective, the circuits constructed and described in this paper demonstrate the development of a circuit structure where the current and voltage at a node may be independently controlled. This result enables the development of alternative circuit structures that can be exploited in constructing more complex circuits, which can be embedded in macro, micro, and mesoscale electrical circuit systems. In addition, because of our results, appropriately designed electrical circuits can be used as testbeds to further explore the properties and range of occurrence of the Braess Paradox in different network systems, including transportation.

Because of the contribution in this paper, there now exists the possibility of further testing the occurrence of the Braess Paradox in controlled laboratory settings, which is not possible in real-world transportation networks. This, we expect, will further enable the understanding of the underlying mechanisms producing the Braess paradox since we now have a reproducible system in the form of an electrical circuit system, which guarantees user-optimized behavior. We hope that this research will also stimulate and advance research in the econophysics of socio-behavioral network systems.

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