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A Bandwidth Auction Mechanism to Enable Affordable Internet Access

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Abstract Although technological developments have provided momentum to extend the frontier of commercially feasible network deployments, the latest data from ITU regarding affordability of ICT services shows that the digital divide between the rich and poor is still an open issue. Therefore, an economic framework is needed to create conditions for affordable network services. In this paper, we propose a set-aside mechanism that can satisfy this need by reserving resources for targeted groups and resolving the practical problem of having greedy users that rationally compete for cheaper resources. In this mechanism, prices are tailored to users' budget capacities. Our simulation results indicate that it is possible to increase the resource allocation for delivering services to the poorest by inducing regular users to compete among themselves.

Keywords Auctions · Discrimination · Selling Bandwidth · Pricing · ICT affordability · Set-aside mechanisms

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1 Introduction

The Millennium Development Goals articulated by the United Nations (UN) include the aim of making the benefits of new technologies available, particularly information and communications, in cooperation with the private sector [1]. Current data indicate that despite the strong progress in Internet penetration from 6% in 2000 to nearly 51% in 2019 [2], a growing gap exists between the adoption rates of rich and poor populations regardless of the ICT price basket monitored [3]. This condition has resulted in "nearly half of people with 4G network coverage remain offline, mainly due to high cost of Internet access". Previous experiences, for instance, of those in rural settings, recognize the necessity of a solid economic framework to balance the social goal of offering services to the poorest and the practical need for incentives for operators to provide services.

Many studies, such as [4]–[7], have shown that auctions are efficient mechanisms for dynamic resource allocation, particularly for selling bandwidth. Their promising results are partially explained because they can handle uncertainty, such as estimations of users' demand functions, and dynamically control allocation through competition between users. Nevertheless, in scenarios where mixed buying-capacity users compete and tight resource constraints are common, it is expected that resources may only be allocated to users with higher budgets.

To counteract this problem, the field of economics has started to study the benefits of subsidized and set-aside auctions for government procurement [8]–[10]. These mechanisms have previously been used to spread allocations favoring selected groups, such as small companies. In subsidy auction schemes, a selected group is granted a proportionally higher bid such that members (i.e., low-budget users) can compete with those in nontargeted groups (i.e., high-budget users). In set-aside auction schemes, resources are preassigned to groups to ensure a minimal, guaranteed allocation within the group, while group members still compete with each other.

Nevertheless, subsidy auction schemes include a number of pitfalls for selling bandwidth. First, it is not clear who is in charge of funding subsidies; second, to have a positive effect, the actual subsidy has to be set, which generally requires a previous study; third, the allocation of units to low-budget groups is not guaranteed; and fourth, this type of auction does not solve the problem of determining users who should be granted the subsidy. Set-aside schemes suffer almost the same problems, but their design offers an allocation guarantee.

The adoption of a set-aside mechanism is complex in practice. Users cannot be easily clustered as belonging to a favored group, and higher budget users have the rational best response of adopting a lower budget role as lower prices are expected. Our work presents a pricing model that is oriented to reach the social goal of providing network access to all while simultaneously addressing these practical challenges. Our pricing strategy introduces a self-selection of users based on budget constraints, enabled by new networking technologies. Specifically, our contributions, sorted by importance, are as follows:

- We present a social-goal-oriented auction mechanism that incentivizes users to reveal private information by providing rewards or assigning penalties, and thereby, reducing the probability of receiving allocated units.
- We present and discuss the trade-off of our auction mechanism between the increased network coverage of low-budget users and the decrease of operator's income.

The remainder of this paper is organized as follows. Section 2 presents a research overview in auction methods for bandwidth allocation as well as the similarities and differences between our approach and previous methods. Next, we present the target network deployment scenario and its main elements. In Section 4, we explain the proposed auction mechanism, starting with a simple auction of two units and three users and then generalize the auction to many users and quantities. In Section 5, we present an alternative to calculate the Nash equilibrium in an actual setting where many users are competing. In Section 6, we discuss the auction setup, introduce other mechanisms implemented for benchmarking, and explain the scenarios used for the simulations. Section 7 presents results that compare our proposed mechanism with other coverage-oriented mechanisms. Finally, Section 8 concludes this paper.

2 Related Work

Dynamic pricing has been intensely studied to enhance network resource allocation; see [11] for a general overview. The present work is close to mechanisms for quality of service (QoS) for inelastic traffic, which are implemented using a call admission control (CAC) function. Similar to many access pricing studies surveyed in [12], we adopt rational users but with uncertain utility functions, where auctions are particularly well suited. In fact, auctions have primarily been used to address the lack of information regarding knowledge of users' behavior and their preferences. Under this setting, the problem is how to allocate scarce resources when QoS is required and the provider wants to pursue a set of goals. In the following, we describe previous work categorized by goals since the main contribution of our proposal is to study the problem under a completely different aim. At the end of the section, we describe those elements that we use from these studies and that help us to develop the presented approach.

With an efficient goal, i.e., to allocate units to users that value them most, a series of auction forms have been developed. Progressive second price (PSP) auctions take advantage of a small message space to sell bandwidth in real time. The mechanism is introduced in [4] for allocating an arbitrarily divisible and additive resource (e.g., bandwidth). To operate, every budget-constrained user submits a two-dimensional bid representing the unit price she is willing to pay and the requested units. Until an equilibrium is reached, users update their bids whenever they receive feedback on others' bids by the auctioneer.

Quantities are allocated to the highest bids, and a winning user pays the social opportunity cost, i.e., the declared willingness to pay for those excluded by the winning user. In this mechanism, the improvement in profits is bounded when sending an untruthful bid, which implies that risk-averse users send their true value. Additionally, the mechanism induces a $\epsilon-$Nash equilibrium with an allocation maximizing the user total value to within $O(\sqrt{\epsilon})$. [13] extends the study to a wide network scenario using independent PSP auctions on each of the links.

Subsequently, [14] verifies PSP's properties in light of a common scenario where users arrive and leave the auction. Under this scenario, the authors found that efficiency is compromised. To overcome this problem in [15], the authors propose a one-shot multi-bid auction scheme, which is related to PSP, for the allocation of a divisible resource in a single link. Important properties, such as incentive compatibility and efficiency, are analytically analyzed and verified. Then, the authors extend the multi-bid auction to a tree topology network [16]. Using a different auction form, [17] introduces the MIDAS mechanism for path pricing. MIDAS runs simultaneous independent per link descending price auctions with the social opportunity cost pricing rule. This rule is used to generate an incentive to bid truthfully. This alternative is shown to be nearly efficient and a revenue maximizer under various network topologies. To summarize, approaches that pursue the efficiency goal allocate units to higher bidders; for this reason, an important property to have is the true valuation revelation.

However, the majority of the previous studies have a complex algorithm as a result of the social opportunity cost pricing rule, which involves solving as many optimal allocation problems as the number of users. The revenue equivalence theorem (see [18]) under a set of conditions, e.g., risk-neutral users, establishes that all auction forms have the same revenue; thus, authors argue that in the search for revenue that complexity is unneeded, and a first price auction form should, therefore, be used. In a first-price auction, users pay their bid and units are allocated to higher bids, which implies that the algorithm simply sorts bids. [19] assumes a scenario in which providers compete and the user (auctioneer) forms a path. Referring to specific topology instances, the authors show that the social opportunity cost pricing rule can lead to customers overpaying the real cost of the cheapest path. To resolve this problem, they introduce three first-price auction forms and derive payment bounds for their $\epsilon-$Nash equilibrium. For a scenario in which users compete, [5] proposes a first-price auction with a single bid to reserve the whole path. The important feature being incorporated is a multiperiod reservation, where providers take the risk of reserving ahead of the entire requested time. A similar work but considering failures during service deployment and providing reimbursement as a compensation is presented in [20]. Aligned with these papers, our work uses a simple algorithm to allocate resources, a second uniform price, but the true valuation revelation property is key for our approach.

In practical scenarios, such as those presented in [7], [21], [22], the provider does not have a unique goal for the auction and there is a trade-off to be addressed. In [21], the trade-off is between congestion reduction and revenue losses. The authors employ a multi-unit Vickery auction at the link level to examine the effect of reserve prices on congestion. The study conceives a private value framework that serves to understand the number of users participating in the auction as the auctioneer increases reserve prices. Using simulations, the authors show that the mean system's delay is reduced with some revenue losses.

We are aware of two studies for maximizing fairness and the operator's profits. The authors in [22] propose an auction with M service levels characterized by a prefixed bandwidth capacity and lower and upper prices proportional to the service-level bandwidth. The allocation and pricing rules are presented, as well as the proof of the system being Pareto optimal. In the case of congestion, operators can downgrade the requested service level unilaterally. A general second-price mechanism (GSP) is proposed in [7] for selling frames (2 ms) in a cellular network context. To operate, users indicate the maximum price that they are willing to pay (e.g., $\$/Mb$), and the cell tower assigns the highest priority to the user with the highest measure, calculated as the price multiplied by the proportional fair value. The price to be paid for that user corresponds to the price of the user with the next highest measure. The process proceeds until capacity is exhausted or there are no more users.

Our work differs from all these alternatives in the aim to reduce the Internet coverage gap between the high-budget (rich) users and low-budget (poor) users. In other words, it is required to spread allocations among users to reach the social goal of providing network access to all. As in [20], [21], we assume a private value framework with a second uniform price rule, where all users pay the bid of the first losing bid. This assumption enables us to maintain the true value revelation property for users bidding in the high-budget auction and to make the allocation algorithm simple. Users are subject to a budget constraint, as in [4]; however, we conceive that there are users with budgets that are completely dissimilar from their valuations, e.g., low-budget users requiring a QoS service. Hence, other problems arise because of the different setting, such as how to maintain providers' revenue while retaining the competition among similar users. Regarding the revenue consideration, this paper is not going to deliver a theoretical revenue loss bound; rather, we use simulations based on real data to provide insights regarding the actual loss.

3 The Target Network Deployment Scenario

For our work, we envision a very general target network scenario, in which a provider offers network services to users and collects payments for their use. In the following, we describe the agents and assumptions.

3.1 Network Agents

Users and providers constitute the interested parties. Users (individuals or organizations) in the coverage area of the provider negotiate resource use through an agent software.

Providers are responsible for establishing end-to-end connections (using third party services if applicable) at the negotiated level of (QoS) and for managing billing and the collection of funds.

The unique characteristic that makes this network scenario different from conventional network deployments is that some users have severe budget constraints. A survey conducted in Ghana, for example, reports $2.50 as the average spending on mobile connectivity per week, with two cents as the reference price per megabyte transferred [23]. In rural and isolated geographic areas, we expect even lower values because the literature describes regions where individuals' income is only one US dollar per day. To achieve universal network access in these areas, it is necessary to favor resource allocations for users with lower budgets at the cost of higher prices for others (e.g., wealthier users and organizations).

3.2 Network Services

To characterize network traffic in our system, we model user interactions at the level of a session to avoid the complexities of modeling traffic at the packet level [24]. In previous studies (cf. [25]), we proposed a series of service level agreements (SLA) to communicate service requests. In this paper, we restrict our attention to the real-time service since models for the other services have already been studied.

4 The Auction Mechanism Design

The results from [25] suggest that dynamic control is extremely important for the target scenario to manage tight resource constraints during congestion. A type of mechanism well-suited for dynamic control is auctioning. Those mechanisms not only perform the control function, but also first discriminate users and, by this way, enlarge the potential of pricing to extract revenues from those high-budget users.

The important issue for reducing the Internet gap is to increase the coverage of low-budget users, which is tightly related to networks' congestion. During non-congested (off-peak) periods, there are enough resources to cover all users' traffic and the price set by auction protocols is in general low, i.e., reserved prices. In contrast, at congested (peak) periods the auction protocol has to increase prices to discourage demand and coverage is then compromised, i.e., in [21]. In fact, units are allocated to those users with the highest valuations and budget, so in peak periods units are frequently allocated to high-budget users. Other auctions protocols, such as those using a second uniform price auction rule, relatively easy to implement, have the same exclusion outcome.

In order to alleviate the exclusion problem, set-aside models have been used exhibiting a positive effect on coverage during congestion. Those models pre-assign a fraction of resources to be sold among the group of low-budget users (L) and the rest among high-budget users (H). As mentioned, during off-peak periods coverage is not a problem, so resource splitting is only considered during peak periods. In particular, set-aside models employ two separate auctions ($\mathcal{L}$ and $\mathcal{H}$) for the sale of pre-allocated units of capacity for users in L and H, respectively. These auctions have reserve prices of p_L and p_H for $\mathcal{L}$ and $\mathcal{H}$, respectively. A fine (D) is charged to winning users in the case of not being able to pay the winning price. For both auctions, the user with the highest bid is served first, followed by the user with the second highest bid and so forth until the supply of units is depleted. All served users pay the highest losing bid or the reserve price if there is not a losing bid or if the losing bid is lower than the reserve price. In the case of a tie, units are randomly allocated.

A particular fact in the scenario being handled is that always the price to be paid in the high budget auction is greater than that in the low-budget auction during congestion. Previous results presented in [25] and theoretical results such as [26] indicate that reserve prices at congestion are higher than the maximal budget available for low-budget users, i.e., $p_H \geq b_L$. Thus, a rational high-budget user able to choose the auction is biased to participate in $\mathcal{L}$.

We want to ensure that high-budget users truthfully participate in auction $\mathcal{H}$ rather than "cheating" and competing in $\mathcal{L}$. In the following, we analyze the high-budget bias by studying the game underlying a set-aside scheme; this study requires some mathematical foundations, which we cover first. Then, using insights from this analysis, we describe the proposed modifications to the auction mechanism to avoid the shortcoming.

4.1 The Assumed Auction Framework for Studying Set-Aside Schemes

We base our work on the private value framework explained in [18]. Under this framework, the information environment is composed of a valuation and budget structure for users (bidders), where values and budgets are private information, and a distribution of information available to those users. Specifically, every user u assigns a value V_u for a unit of bandwidth – the maximum amount that user u is willing to pay for the unit. V_u is independent and identically distributed on interval $[0, 1]$ according to the increasing accumulated distribution function F_V. Additionally, user u is subject to her budget – the maximum amount that user u is able to cover for a bandwidth unit. The budget for a user $u \in L$ follows an independent and identically distributed W_{L_u} on some interval $[0, b_L]$ according to the increasing distribution function F_{W_L}. The budget for a user $u \in H$ follows an independent and identically distributed

W_{H_u} on the interval $[b_L, b_H]$ according to the increasing distribution function F_{W_H}, with $0 < b_L < b_H \leq 1$. It is assumed that all of the previous distribution functions F admit a continuous density $f \equiv F^{'}$.

A user $u \in L$ knows the realizations v_u and w_u of V_u and W_{L_u}, respectively; likewise, a user $u \in H$ knows the realizations v_u and w_u of V_u and W_{H_u}, respectively. Users are aware that other users' values are independent and identically distributed according to F_V and that their budgets are independent and identically distributed according to F_{W_H} when users are high-budget users and according to F_{W_L} when users are low-budget users. Valuations and budgets are assumed to be independent. A user u enrolls in auctions by sending a bid β_u, which is called the *strategy* of the user in the auction. Users attempt to optimize their expected profits, and, for now, all other parts of the model are known by the users; thus, the distribution and the number of participating users are common knowledge.

To understand the user's optimal behavior when bidding, we first analyze users' possible actions, i.e., strategies, and the information that are revealing. First, because low-budget users are constrained by the value b_L, a high-budget user sending a bid greater than b_L is revealing that she belongs to H. Second, low-budget users only participate in $\mathcal{L}$ by sending $\beta_u = min\{v_u, w_u\}$. This result is the conclusion of the optimal bid for a user truly bidding, i.e, bidding in the auction created for the group that she actually belongs, see proposition 8 (A) and the fact that $p_H \geq b_L$, which is implying that low-budget participating in $\mathcal{H}$ have a maximum pay-off of zero.

Third, a high-budget user u has two strategies, i.e., to compete in $\mathcal{H}$ or in $\mathcal{L}$. However, the user only competes in $\mathcal{H}$ when her private realization v_u is greater than b_L because, like a low-budget user, her maximum pay-off is zero when enrolling in $\mathcal{L}$. Therefore, from the auctioneer's perspective, a high-budget user with $v_u \leq b_L$ has the same behavior as a low-budget user, and we have an extended set of low-budget users $L^{'} = L \cup \{u \in H | v_u \leq b_L\}$ and a reduced set of high-budget users $H^{'} = \{u \in H, v_u \geq b_L\}$. A high-budget user u with $v_u \geq b_L$ competing in $\mathcal{L}$ maximizes her expected payoff sending $\beta_u = b_L$, see proposition 9 (A). To summarize, a user $u \in H^{'}$ has two strategies $s_L = (\mathcal{L}, b_L)$ and $s_H = (\mathcal{H}, \beta_u)$, where β_u is defined in proposition 8 (A) and users in $H - H^{'}$ should be modeled as a low-budget user.

Figure 1 shows the framework being used on both auctions when defining the following random variables for users' bids and their relative order. We define as $X_L = \min\{V, W_L\}$ the minimum of the random variables for the value and budget. Observe that low-budget user bids are distributed according to X_L. Also, we denote as Y_n^c the c highest order statistic of n random variables X_L; in other words, Y_n^c is the random variable that represents the c-highest bid, and by $F_{Y_n^c}$ the distribution function of Y_n^c. Assuming c_L units of capacity for sale and N_L users competing in $\mathcal{L}$ for a unit, the expected payment of a low-budget user u winning the auction is:

$$F_{Y_{N_L-1}^{c_L}}(\beta_u) E[Y_{N_L-1}^{c_L} | Y_{N_L-1}^{c_L} \leq \beta_u] \tag{1}$$

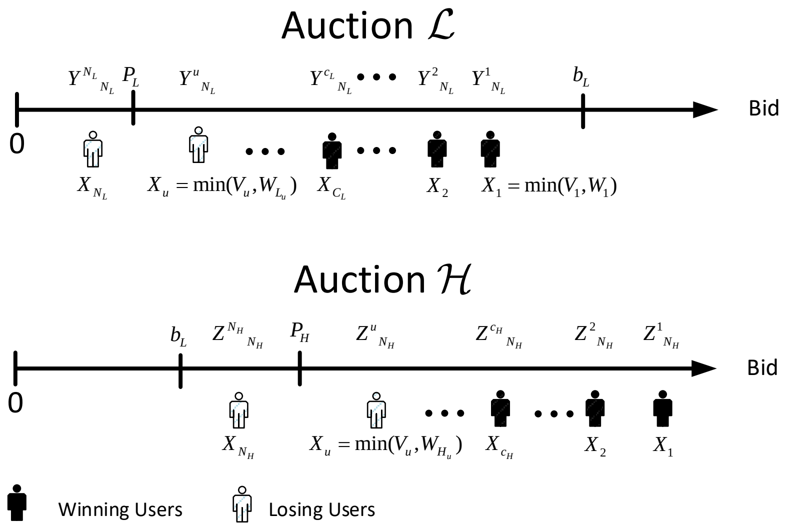

Fig. 1 The two auction framework

where $F_{Y^{c_L}_{N_L-1}}(\beta_u)$ is the probability of winning with bid β_u, i.e., the bid β_u is greater than c_L other bids; and $E[Y^{c_L}_{N_L-1}|Y^{c_L}_{N_L-1} \leq \beta_u]$ represents the expected c_L highest bid among the remainder of the users in $\mathcal{L}$ given that β_u is not less.

In a similar way, but for high-budget users, we define $X_H = \min\{V, W_H\}$, Z^c_n as the c highest order statistic of n random variables X_H, and by $F_{Z^c_n}$, the distribution function of Z^c_n. Assuming c_H units for sale and N_H users competing, the expected payment for that user is $F_{Z^{c_H}_{N_H-1}}(\beta_u)E[Z^{c_H}_{N_H-1}|Z^{c_H}_{N_H-1} \leq \beta_u]$. Note that X_H stochastically dominates X_L by definition.

Henceforth, we consider the group of high-budget users H as those in $H^{'}$ and the remainder of the users as low-budget users L, and we denote by N_H and N_L the respective number of users in these groups.

4.2 The Underlying Set-Aside Auction Game

We start by assuming a game with $N_H + 1$ high-budget users and N_L low-budget users and two units of capacity to allocate, one per auction. Every user requires a unit of capacity. We use the superscript $(^+)$ to indicate the game induced by the two set-aside auctions. Under this setting, we analyze as a first step the pure Nash equilibrium to conclude that always at least a high-budget user enrolls in $\mathcal{L}$ and excludes low-budget users. After, we analyze how this conclusion varies under the more general setting where users decide their strategies probabilistically, which corresponds to a Nash equilibrium for mixed strategies, and introduce important properties of the expected payoff functions. Finally, we analyze the resulting equations to give the insights that we used to construct the proposed mechanism.

4.2.1 Pure Strategies

We discuss pure Nash equilibria for this game in terms of high-budget users. Denote by $\mu_u^+(s, n_H)$ the expected payoff for user $u \in H$ when using strategies $s = \{s_H, s_L\}$ and n_H other users participate in auction $\mathcal{H}$. On one hand, the user u can employ the strategy s_H. The payoff for this case depends on the number of users competing in $\mathcal{H}$. In fact, there are two cases: (1) when $n_H > 0$, the user expects her profits to be equal to the difference between the valuation and the highest losing bid in case of winning; in other words, the payoff for this case is equal to the probability of winning when bidding β_u multiplied by the difference between her valuation and the expected first order statistic from the rest of n_H users participating; and (2) when $n_H = 0$, the user is the only one competing in $\mathcal{H}$. In this case the price to paid is then the reserve price p_H. The following equation shows the expression for both cases:

$$\mu_u^+(s_H, n_H) = \begin{cases} F_{Z^1_{n_H}}(\beta_u)(v_u - E[Z^1_{n_H} | Z^1_{n_H} \leq \beta_u]) & , n_H > 0 \\ v_u - p_H & , n_H = 0. \end{cases} \tag{2}$$

On the other hand, when using strategy s_L, the high-budget user wins the unit if she is the only high-budget user participating in $\mathcal{L}$ and is randomly allocated the unit when other high-budget users are also competing in $\mathcal{L}$. Observe that by bidding b_L, the user $u \in H$ dominates all other low-budget users. Therefore, the payoff of using strategy s_L is equal to the spread between the user u valuation and the first order statistic of all low-budget users. The following equation represents mathematically the strategy payoff when $N_H - n_H$ high budget users compete in $\mathcal{L}$.

$$\mu_u^+(s_L, n_H) = \frac{v_u - E[Y^1_{N_L}]}{N_H - n_H}. \tag{3}$$

Under this setting, low-budget users are completely excluded and their bids are only used to establish the winning price ($E[Y^1_{N_L}]$). However, the unit is not always allocated to user u due to other high-budget users participating in auction $\mathcal{L}$. In that case, the random allocation rule is applied, and user u has probability $\frac{1}{N_H - n_H}$ of winning the unit. For $n_H = N_H - 1$, we obtain the maximum payoff possible from 3 and its minimum when $n_H = 0$. In Table 1, we present an example of the payoffs for the case of two high-budget users.

From Table 1, we can calculate the three possible Nash equilibria ((s_L,s_H), (s_H,s_L), and (s_L,s_L)) that depend on $\frac{1}{2}(v_u + E[Y^1_1]) \geq p_H, \forall u \in H$. Thus, the desired behavior of all high-budget users choosing the auction $\mathcal{H}$ it is not a pure Nash equilibrium, and consequently, splitting resources is not sufficient to guarantee a low-budget allocation.

Table 1 Two-auction game's payoffs for high-budget users

		User 2: s_H	User 2: s_L
User 1	s_H	$\mu_1^+(s_H, 1) = F_{Z_1^1}(\beta_1)(v_1 - E[Z_1^1 \mid Z_1^1 <= \beta_1])$, $\mu_2^+(s_H, 1) = F_{Z_1^1}(\beta_2)(v_2 - E[Z_1^1 \mid Z_1^1 <= \beta_2])$	$\mu_1^+(s_H, 0) = v_1 - p_H$, $\mu_2^+(s_L, 1) = v_2 - E[Y_1^1]$
	s_L	$\mu_1^+(s_L, 1) = v_1 - E[Y_1^1]$, $\mu_2^+(s_H, 0) = v_2 - p_H$	$\mu_1^+(s_L, 0) = \frac{1}{2}(v_1 - E[Y_1^1])$, $\mu_2^+(s_L, 0) = \frac{1}{2}(v_2 - E[Y_1^1])$

4.2.2 Mixed Strategies

We now study the Nash equilibria with mixed strategies. For this analysis, we restrict our attention to non-averse risk users. They assume that there exists a common probability $0 \le q < 1$ that a user $u \in H$ plays strategy s_L. The expected payoff function for a user is modelled as the weighted sum over $s_H \in \{0, N_H\}$ of the payoffs when s_H other users participate in $\mathcal{H}$ given by equations 2 and 3 for strategies s_H and s_L, respectively. The weights correspond to the probability of the binomial experiment consisting in s_H users maintain in $\mathcal{H}$ from N_H other users playing s_L with probability q. The following states the equation representing the expected payoff for user u when playing strategy s_H – observe that N_H excludes user u–:

$$\pi_u^+(s_H, q) = \sum_{k=0}^{N_H} B(k, N_H, q)\mu_u^+(s_H, N_H - k), \tag{4}$$

where $B(k, N_H, q)$ is the binomial distribution.[1] $B(k, N_H, q)$ represents the probability that $N_H - k$ users remain in auction H of the N_H other users. Likewise, the expected payoff of strategy s_L is given by:

$$\pi_u^+(s_L, q) = \sum_{k=0}^{N_H} B(k, N_H, q)\mu_u^+(s_L, N_H - k). \tag{5}$$

Figure 2 plots the expected payoff functions for the special setting of two high-budget users. Note that, for this setting, $\pi_u^+(s_H, q)$ is continuous, linear and non-decreasing with respect to q and that $\pi_u^+(s_L, q)$ is continuous, linear and non-increasing in q. A Nash equilibrium in mixed strategies exists at the point where the lines $\pi_u^+(s_H, q)$ and $\pi_u^+(s_L, q)$ intersect for both users. Then, an existance condition is simply that $v_u + E[Y_{N_L}^1] \ge 2p_H, \forall u \in \{1, 2\}$. We denote by $q^+ = (q_1^+, q_2^+)$ the Nash equilibrium, which is shown in the figure.

We now prove that these properties still remain when more users are competing on $\mathcal{H}$ and use them to provide a more general condition for the existance of a Nash Equilibrium in mixed strategies. We start with the following proposition, which states that the high-budget payoff (see equation 2) when using strategy s_H is increasing as less users are competing with her in the same auction n_H. From equation 2, we can say that the high-budget payoff is decreasing with $N_H - n_H$.

[1] $B(k, N_H, q) = C(k, N_H)q^k(1 - q)^{N_H - k}$ with $C(k, n)$, the k combination of n users.

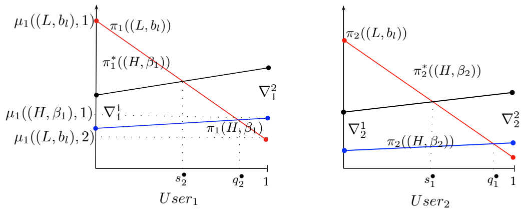

Fig. 2 Users' expected payoffs for the set-aside auction game. Y axis: payoff. X axis: probability

Proposition 1 *The expected payment $\mu_u^+(s_H, n_H)$ is a non-increasing function of the number of users in auction $\mathcal{H}$, which is equivalent to the following inequality for any n_{H_1}, n_{H_2} with $1 \leq n_{H_1} < n_{H_2} \leq N_H$*

$$E[Z^1_{n_{H_1}} | Z^1_{n_{H_1}} \leq \beta_u] \leq E[Z^1_{n_{H_2}} | Z^1_{n_{H_2}} \leq \beta_u]. \tag{6}$$

moreover, because $v_u - E[Z^1_{n_H} | Z^1_{n_H} \leq \beta_u] \geq 0$ and $F_{Z^1_{n_{H_1}}}(\beta_u) \geq F_{Z^1_{n_{H_2}}}(\beta_u) \geq 0$, we can conclude by applying proposition 1:

$$1 \leq n_{H_1} < n_{H_2} \leq N_H \Rightarrow \mu_u^+(s_H, n_{H_1}) \geq \mu_u^+(s_H, n_{H_2}). \tag{7}$$

In the following proposition, we study the properties of $\pi_u^+(s_H, q)$ and $\pi_u^+(s_L, q)$. This proposition is saying that as long as a user expects that other users increase their probability of using strategy s_L, then when using strategy s_H the expected payoff is greater and when using strategy s_L the expected payoff is lower. Therefore, if there is a probability q where both functions are equal, she is indifferent between playing strategy s_H or playing strategy s_L.

Proposition 2 *$\pi_u^+(s_H, q)$ is a non-decreasing function, and $\pi_u^+(s_L, q)$ is a non-increasing function of q*

Now we use the properties studied so far to provide the required conditions for the existence and uniqueness of a Nash equilibrium in mixed strategies, which is stated in the following proposition.

Proposition 3 *If $\frac{N_H v_u + E[Y^1_{N_L}]}{N_H + 1} \geq p_H, \quad \forall u \in H$, then there exists at least a Nash equilibrium in mixed strategies for the two-auction game, which is denoted by q^+.*

We intentionally plot Figure 2 with q_1^+, q_2^+ probabilities close to 1. By competing in $\mathcal{L}$, a high-budget user decreases the number of opponents – only competes with other high-budget users – and the winning price is lower than that in $\mathcal{H}$. This is the major shortcoming of applying a set-aside scheme in the presence of information asymmetries. Indeed, this setting makes the payoff of strategy s_L linearly decreasing in n_H by the winning probability. Therefore, to guarantee the low-budget allocation by discouraging s_L, it is necessary to introduce penalties by reducing its payoff or its winning probability.

4.2.3 Alternatives for discouraging high-budget users employing s_L strategy

We investigate two approaches to maintain high-budget users competing in $\mathcal{H}$. The first approach is to encourage attachment to auction $\mathcal{H}$ by granting a reward R to users participating in auction $\mathcal{H}$. Combining equations 2 and 3, the reward R must satisfy $\forall u \in H$:

$$F_{Z^1_{N_H}}(\beta_u)(v_u - E[Z^1_{N_H}|Z^1_{N_H} \leq \beta_u] + R) \geq v_u - E[Y^1_{N_L}],$$

which means that:

$$F_{Z^1_{N_H}}(\beta_u)(v_u + R) \geq v_u - E[Y^1_{N_L}] + F_{Z^1_{N_H}}(\beta_u)(E[Y^1_{N_L}])),$$
$$R \geq (v_u - E[Y^1_{N_L}])\frac{1-F_{Z^1_{N_H}}(\beta_u)}{F_{Z^1_{N_H}}(\beta_u)}.$$

The former equation relates the reward R with the winning probability. Note that whenever $F_{Z^1_{N_H}}(\beta_u) \to 1$, the reward tends to 0. Thus, a user with a high budget and high valuation does not require any incentive and always remains in $\mathcal{H}$. Conversely, whenever $F_{Z^1_{N_H}}(\beta_u) \to 0$, the reward tends to infinity. Therefore, a user with a low budget or valuation almost always competes in auction $\mathcal{L}$. These observations appear to indicate that rewards are better for the first group and that penalties might work for the second.

The second approach is to reduce the probability of being assigned a unit in the auction $\mathcal{L}$. Due to the assignment rule employed, high budget-users using strategy s_L have only competition from other high-budget users. So to reduce the probability, it is required to involve in the assignment rule all low-budget users. In particular, if we use a lottery for allocating units in the $\mathcal{L}$, the denominator value in 3 is rapidly increased and the payoff is reduced.

The proposed mechanism starts rewarding truthful users with a high budget and valuation and reduces untruthful users' winning probabilities. Rewards are in the form of allowing users to pay those prices charged for the low-budget set, and lowering the probability is accomplished by using a random allocation rule in the low-budget set. Thus, users that are "cheating" have no choice but to compete with more users.

Moreover, recall from the definition of equilibrium in mixed strategies that a user $u \in H$ is indifferent between playing s_H and s_L when other users play s_L with probability q^+_u. Therefore, if a unique probability q^+ applies for all users, then they can delegate to a trustful system the auction decision. In such a case, they can agree with the system to put them to compete in $\mathcal{L}$ with probability q^+. We use this last fact and make the users to act truthfully by guaranteeing that users sending $\beta_u > b_L$ are promoted to auction $\mathcal{L}$ with the calculated Nash equilibrium probability (q^+).

4.3 The Winning Probability Reduction Mechanism (RwP)

Formally, we state that a user is an untruthful user when despite having a sufficient budget and valuation to compete within the H auction group she decides to go to the L group. The following rules apply for peak intervals:

- Two parallel auctions occur for high and low budget users.
- A second uniform price auction for high-budget users and a random allocation among users competing in the L group takes place.
- Users can send bids to any auction.
- Users in auction $\mathcal{L}$ who have been allocated a unit pay the reserve price for the group, p_L. Winning users in auction $\mathcal{H}$ are charged p_L with probability Q and the $\mathcal{H}$ auction clearing price with probability $(1-Q)$.

For off-peak periods, a single second uniform price auction is used. [2]

4.3.1 Mixed Nash Equilibria

We use the superscript $(^\circ)$ to indicate the game induced by the RwP and refer to it as the RwP game. This game has the same users and strategies as the two set-aside auctions game presented, but it has different payoffs for both strategies. When there are n_H users participating in $\mathcal{H}$, the expected payoff $\mu_u^\circ(s_H, n_H, Q)$ is defined as:

$$\mu_u^\circ(s_H, n_H, Q) = \begin{cases} \mu_u^+(s_H, n_H) + \Delta_u^\circ(s_H, n_H, Q), \; n_H > 0 \\ \mu_u^+(s_H, n_H) + \Delta_u^\circ(s_H, 0, Q), \quad n_H = 0, \end{cases} \tag{8}$$

where $\Delta_u^\circ(s_H, n_H, Q) = QF_{Z^1_{n_H}}(\beta_u)(E[Z^1_{n_H} | Z^1_{n_H} \leq \beta_u] - p_L])$ and $\Delta_u^\circ(s_H, 0, Q) = Q(p_H - p_L)$. The mechanism reduces the winning probability when using the s_L strategy. As a user is randomly allocated the unit between all users in auction $\mathcal{L}$, when there are n_H users participating in auction $\mathcal{H}$, the s_L strategy's payoff is:

$$\mu_u^\circ(s_L, n_H, Q) = \frac{v_u - p_L}{N_L + (N_H - n_H + 1)}. \tag{9}$$

From these equations, we can restate user u's expected payoff for the $s \in \{s_H, s_L\}$ strategy when she assumes a probability q that all other users employ the s_L strategy,

$$\pi_u^\circ(s, q, Q) = \sum_{k=0}^{N_H} B(k, N_H, q) \mu_u^\circ(s, N_H - k, Q). \tag{10}$$

High-budget users bidding in auction $\mathcal{H}$ obtain better payoffs than the two set-aside auctions game – they can pay p_L. The potential problem is the

[2] Note that the possible shill bidding strategy of sending many requests to the $\mathcal{L}$ auction can be addressed in the auction software. In fact, users have to specify a flow description, which includes source and destination IP addresses, for the connection, so that the software may implement an upper bound for resource requests with the same flow description.

revenue for the provider. To discourage high-budget users from being untruthful, we decrease providers' income from low-budget users; thus, it is required to understand how much revenue is affected by its implementation, which is studied through simulations in section 7.

Following the same steps for the two auction game, we assume that $N_H + 1 \geq 3$ high-budget users and N_L low-budget users are requesting a unit of capacity and show that functions $\pi_u^\circ(s_L, q, Q)$ are decreasing in q and that $\pi_u^\circ(s_H, q, Q)$ are increasing in q. Therefore, we can extend the required conditions given in equation 12. In this case, a mixed Nash equilibrium corresponds to an N_H-tuple such that every user is indifferent between both strategies. The element u in the tuple corresponds to user u's probability of playing strategy s_L.

To start, we study the behavior of the payoff for a high-budget user in the case of the RwP mechanism.

Proposition 4 *$\mu_u^\circ(s_H, n_h)$ is a decreasing function of the number of users in auction $\mathcal{H}$, which is equivalent to the following inequality for any n_{h_1}, n_{h_2} with $1 \leq n_{h_1} < n_{h_2} \leq N_H$*

$$\mu_u^\circ(s_H, n_{h_1}) \geq \mu_u^\circ(s_H, n_{h_2}). \tag{11}$$

Following the same steps as in Proposition 2, we conclude that $\pi_h^\circ(s_H, q)$ is a non-decreasing function of q. Let q° denote the Nash equilibrium in mixed strategies for the RwP. A mixed Nash equilibrium exists with $q_u^\circ < 1$, only if $\forall u \in H$:

$$\frac{v_u(N_L+1) + p_L(1-(N_L+2)Q)}{(N_L+2)(1-Q)} \geq p_H. \tag{12}$$

Hence, RwG has a unique mixed Nash equilibrium [3].

If $Q = 0$, we have the same condition as with the two-auction game; for $Q \to 1$, the condition is always true as long as $v_u \geq E[Y_1^1], \forall u \in H$. Thus, there is always a mixed strategy with $q_u^\circ < 1$.

The modified game allows the probability of a high-budget user going into the $\mathcal{L}$ auction to be reduced, i.e., $(q_u^+ > q_u^\circ), \forall u \in H$; see Figure 2. For the entire system, the game creates a trade-off between offering rewards (Q) and the equilibrium probability reduction $(q_u^+ - q_u^\circ)$. To measure this trade-off, we introduce the formula $z_{tr} = \frac{\sum_{h=1}^{2} q_h^+ - q_h^\circ}{1 \cdot Q}$, which represents the ratio between the expected number of users induced to remain in the H auction against the expected number of users that receive the reward. This formula will be used to calculate parameter Q.

4.4 Extension for M units

Denote by c_L, c_H the number of available units for $\mathcal{L}$ and $\mathcal{H}$ auctions. Let us denote by $e_L^{c,n} = E[Y_n^c]$ and by $e_H^{c,n}(\beta_u) = E[Z_n^c | Z_n^c \leq \beta_u]$. User u's payoff

[3] The reader can see that $\pi_h^\circ(s_L, q)$ is decreasing by applying the same steps as in Proposition 10.

$$\mu_u^{\circ}(s_H, n_H) = \begin{cases} v_u - Qe_L^{1,N_L} - (1-Q)p_H & \text{if } n_H - c_H \le 0 \wedge \overline{c_L} \le 0 \\ v_u - Qe_L^{\overline{c_L},N_L} - (1-Q)p_H & \text{if } n_H - c_H \le 0 \wedge \overline{c_L} > 0 \\ \mu_u(s_H, n_H) + \Delta_u^{\circ 1}(s_H, n_H, Q) & \text{if } n_H - c_H > 0 \wedge \overline{c_L} \le 0 \\ \mu_u(s_H, n_H) + \Delta_u^{\circ 2}(s_H, n_H, Q) & \text{if } n_H - c_H > 0 \wedge \overline{c_L} > 0 \end{cases}, \tag{15}$$

under the two side-aside auctions game when playing strategy s_H and n_H other users are participating in auction $\mathcal{H}$ is:

$$\mu_u^{+}(s_H, n_H) = \begin{cases} v_u - p_H & , n_H - c_H \le 0 \\ F_{Z_{n_H}^{c_H}}(\beta_u)(v_u - e_H^{c_H,n_H}(\beta_u)) & , n_H - c_H > 0 \end{cases}. \tag{13}$$

The first part of this equation $n_H - c_H \le 0$ occurs when there are not enough users to deplete the capacity c_H and they pay the reserve price p_H. The second part is the expected value gain by the user minus the expected payment $(e_H^{c_H,n_H}(\beta_u))$ when β_u is greater than the c_H highest order statistic.

On the one hand, more users $(N_H - n_H)$ can decide to participate in $\mathcal{L}$ than the available capacity c_L; hence, they have to pay the highest order statistic calculated over the low-budget users, but units are randomly allocated between them with probability $\frac{c_L}{N_H - n_H + 1}$ as capacity is not sufficient. On the other hand, there are enough units $c_L > N_H - n_H$ such that users have to pay the $c_L - (N_H - n_H)$ highest order statistic over the N_L low-budget users. For this case, they always win a unit. Defining $\overline{c_L} = c_L - (N_H - n_H)$, user u's payoff for s_L strategy under the two set-aside auctions game is given by:

$$\mu_u^{+}(s_L, n_H) = \begin{cases} \frac{c_L(v_u - e_L^{1,N_L})}{N_H - n_H + 1} & \text{if } \overline{c_L} \le 0 \\ (v_u - e_L^{\overline{c_L},N_L}) & \text{if } \overline{c_L} > 0 \end{cases}, \tag{14}$$

the payoff functions for strategy s_H under the RwG game are derived from the number of users participating in $\mathcal{H}$, but in this case, it makes changes to the expected payment in both auctions. Therefore, the payoff for strategy s_H has four cases shown in equation (15), where

$$\Delta_u^{\circ 1}(s_H, n_H, Q) = QF_{Z_{n_H}^{c_H}}(\beta_u)(e_H^{c_H,n_H}(\beta_u) - e_L^{1,N_L})$$
$$\Delta_u^{\circ 2}(s_H, n_H, Q) = QF_{Z_{n_H}^{c_H}}(\beta_u)(e_H^{c_H,n_H}(\beta_u) - e_L^{\overline{c_L},N_L}).$$

The RwG payoff function when using strategy s_L is

$$\mu_u^{\circ}(s_L, n_H) = \frac{(v_u - p_L)c_L}{(1 - F_{X_L}(p_L))N_L + N_H - n_H + 1}. \tag{16}$$

In general, $\mu_u^{+}(\cdot)$ maintains the same behavior as in the one unit scenario; $\mu^{+}(s_L, \cdot)$ is decreasing and $\mu^{+}(s_H, \cdot)$ is increasing with respect to $N_H - n_H$. However, additional behaviors are created by the auction's quantities for s_L and s_H. For s_L and $N_H - n_H \in [0, c_L]$, $\mu_u^{+}(s_L, \cdot)$ decreases because of the increase in auction prices between low-budget users. For $N_H - n_H > c_L$, as before, the function decreases as a consequence of the lower probability related to the random allocation. $\mu_u^{+}(s_H, \cdot)$ is constant in the interval $n_H \le c_H$, where users pay the reserve price p_H. Applying Proposition 1, we can conclude that $\pi_u^+(\cdot)$ have the same properties as the one unit case; thus, we have a mixed Nash equilibrium with $q_u^+ < 1, \forall u \in H$ whenever:

$$\frac{(N_h + 1 - c_L)v_u - c_L e_L^{1,N_L}}{N_h + 1} \geq p_H. \tag{17}$$

$\mu_u^\circ(s_H, \cdot)$ is always increasing as the reward is fixed regarding the number of users participating in $\mathcal{L}$. Again, we can apply Proposition 1 to conclude that the $\pi_u^\circ(\cdot)$ have the same properties as the one unit case; thus, there exists a mixed Nash equilibrium with $q_u^\circ < 1, \forall u \in H$ whenever $\mu^\circ(s_H, 0) \geq \mu^\circ(s_L, 0)$.

5 On a Practical Approach to Calculate Nash Equilibria

In the previous section we described the framework used to model a set-aside auction design that promotes resource distribution among users considering their budget constraints. From that study, we concluded that a canibalization problem previously commented in the revenue management literature [27] is present, and therefore, adjustments to that model are needed. As an alternative, we introduce the *RwP* mechanism to avoid the canibalization problem and verified the conditions for existence and uniqueness of the Nash equilibria. In this section, we investigate the related problem of how to calculate the Nash equilibrium in an actual setting where many users are competing and a regulated provider wants to offer a limited bandwidth for them.

At the first subsection, we consider the case of a unit being offered, e.g., a Gbps channel being assigned for a specified period of time and users with uniform valuations and budgets. For this particular setting, we develop analytic forms for payoff functions, which are used to calculate the Nash Equilibrium. This approach, however, is limited as former functions depend on factorial terms that make them intractable to calculate as long as the number of users increases. To couple with that problem, we approximate the binomial distribution by a normal distribution and expected payments by results from extreme value theory. All of these developments let us calculate equilibrium probabilities used to enforce users to employ as their optimal strategy for the one revealing their true group assignment.

In the second part, we extend the model for any number of units. We are interested in the expected value of the $N_H - c_H$ maximum bid from a sample of N_H bids given a bid β_u and its corresponding winning probability. Intermediate order statistics have been used in the literature to establish bounds and limit distributions for those values whenever a large user population is present. We employ the results from this literature to calculate the Nash equilibrium verifying the required conditions (7) under the uniform assumption. The reader should also observe that more general resulting distribution functions, such as, those with an exponential parent, satisfy both the von Mises and derivative conditions and, therefore, are extremable.

Distribution	Auction	W_i	V_i	density (f_X)	limit distrib ($G_i(y,n_i)$)
Uniform	$\mathcal{L}$	$[0, b_L]$	$[0, 1]$	$\begin{cases} 0 & y < 0 \\ \frac{b_L+1-2y}{b_L} & 0 \le y \le b_L \\ 1 & y > b_L \end{cases}$	$\begin{cases} 0 & y < 0 \\ \exp\left(\frac{y-b_l}{a_{n_l}}\right) & y \le b_L \\ 1 & y > b_L \end{cases}$
	$\mathcal{H}$	$[b_L, 1]$	$[b_L, 1]$	$\begin{cases} 0 & y < b_L \\ \frac{2-2y}{(1-b_L)^2} & b_L \le y \le 1 \\ 1 & y > 1 \end{cases}$	$\begin{cases} 0 & y < b_L \\ \exp -\left(\frac{1-y}{a_{n_h}}\right)^2 & b_L \le y \le 1 \\ 1 & y > 1 \end{cases}$

Table 2 Distribution functions used and their corresponding limit distribution function, $b_H = 1$.

The last two subsections are devoted to resolving two practical issues to be addressed in order to operate the mechanism: (1) how to determine a unique equilibrium probability for users in the $\mathcal{H}$ and (2) how to define an optimal Q, so that the provider achieves the optimal threshold between the cost of offering incentives and its corresponding decrease in the high-budget s_L strategy employment.

5.1 The Special Case of a Capacity Unit Per Auction

From now on, we assume that users' valuations and their budgets are uniformly distributed according to the parameters in Table 2. From the optimal bidding strategy already discussed in the previous section, we know that the bid will be the minimum between the budget and the valuation, so that the bid distribution for each group is the minimum of these two distribution functions. The distribution densities are shown in Table 2 under the column density (f_X). To calculate the Nash equilibria, we need to calculate π_u^+, π_u°, by using equations (4), (5), (10) and these functions depend on the expected payoff $E[Y_n^1(y)]$ and $E[Z_n^1(y)|Z_n^1(y) \le \beta_u]$ for each of the auctions. The following proposition presents the analytical formula resulting from replacing bid distributions on table 2 into the equations defining these values.

Proposition 5 *The expected value of the n order statistic from a sample of n identically, independently distributed random variables X_L with uniform parents is given by the formula:*

$$E[Y_n^1(y)] = b_L - \sum_{i=1}^{n} \frac{(n!)^2 b_L^i}{(n+i)!(n+1-i)!} - \frac{(n!)^2 b_L^{n+1}}{(2n+1)!};$$

likewise, the expected conditional value for the n order statistic from a sample of n identically, independently distributed random variables X_H is given by the formula:

$$E[Z_n^1(y)|Z_n^1(y) \le \beta_u] = \beta_u - \frac{b_L^{n+1}(n+1)^{-1} + \overline{I}(n)}{(F_{X_H}(\beta_u))^n}.$$

To calculate the Nash equilibrium, we can employ any gradient Newton algorithm by employing the expression for the winning probability $F_{X_H}^{N_H}$ and the former two equations until we find a q° such that $\pi_u^\circ(s_L, Q, q) - \pi_u^\circ(s_H, Q, q) = 0$. However, this approach is only computationally tractable as long as n_H and n_L are small numbers and, therefore, it is required another approach to calculate previous values. Here, we assume conditions for approximating the binomial distribution by the normal distribution; therefore, $\pi_u(\cdot, q)$ is approximately:

$$\pi_u^+(\cdot, q) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} exp(-x^2)\mu_u^+(\cdot, n(x))dx \tag{18}$$

with $n(x) = \lceil N_H(1-q) - x\sqrt{2N_H q(1-q)}\rceil$ and $\mu_u^+(\cdot, n(x)) = \mu_u^+(\cdot, 0)$ for $n(x) < 0$ and $\mu_u^+(\cdot, n(x)) = \mu_u^+(\cdot, N_H)$ for $n(x) > N_H$. Furthermore, the integration can be evaluated using the Gauss-Hermite formula:

$$\begin{aligned} \pi_u^+(\cdot, q) &= \tfrac{1}{\sqrt{\pi}} \textstyle\int_{-\infty}^{\infty} exp(-x^2)\mu_u^+(\cdot, n(x))dx \\ &\cong \tfrac{1}{\sqrt{\pi}} \textstyle\sum_{j=1}^{m} \alpha_j \mu_u^+(\cdot, n(x_j)), \end{aligned} \tag{19}$$

where α_j and x_j are the j weight factor and zero, respectively, of the $(2m-1)$th-order Hermite polynomial[4].

We are now interested in calculating μ^+ and μ° such that we can compute (19). The following proposition allows us to have bounds in the auctions' expected payments.

Proposition 6 *The functions* $\frac{1}{F_{X_H}}$, $\frac{1}{F_{X_L}}$, $\frac{1}{1-F_{X_H}}$, *and* $\frac{1}{1-F_{X_L}}$ *are convex.*

The following inequality is valid by Theorem 4.4.9 in [28](pp:79) and Proposition 6 applied to the random variable F_{X_L}

$$F_{X_L}^{-1}\left(\frac{N_L - 1}{N_L}\right) \leq E[Y_{N_L}^1] \leq F_{X_L}^{-1}(1) = b_L, \tag{20}$$

where $F_{X_L}^{-1}$ represents the inverse of the distribution function F_{X_L}. Additionally, note that $E[Z_{n_H}^1 | Z_{n_H}^1 \leq \beta_u]$ is equal to the expected value from the truncated distribution $(F_{X_H})^{n_H}$ at β_u. Proposition 6 can be verified for this truncated function; thus:

$$F_{X_{H_{\beta_u}}}^{-1}\left(\frac{n_H - 1}{n_H}\right) \leq E[Z_{n_H}^1 | Z_{n_H}^1 \leq \beta_u] \leq F_{X_{H_{\beta_u}}}^{-1}(1) = \beta_u. \tag{21}$$

where $F_{X_{H_{\beta_u}}}^{-1}$ represents the inverse of the distribution function F_{X_H} truncated at β_u. These results imply that $Y_{N_L}^1$ and the truncated version of $Z_{n_H}^1$ are uniformly integrable.

In the following proposition, we use limit distributions [29] to approximate $(F_{X_H}(\beta_u))^{n_H}$, the probability of winning the unit for a bid β_u, when $n_H \to \infty$.

[4] We use a polynomial of order 20 to calculate this equation, which yields an error of less than 10^{-7}.

Proposition 7 *(von Mises conditions) Distributions F_{X_L} and F_{X_H} are extremable of the Weibull family. Furthermore, the exponents are $\alpha = 1$ for X_L and $\alpha = 2$ for X_H with standardizing constants:*

$$a_n = F_{X_i}^{-1}(1) - F_{X_i}^{-1}\left(1 - \frac{1}{n}\right), \tag{22}$$

where $F_{X_i}^{-1}$ represents the inverse of the distribution function F_{X_i} with $i \in \{L, H\}$ (see Table 2 under the G_i).

5.2 Multiple Units Per Auction

For deriving limit distributions of Y_n^c, Z_n^c such that the Nash equilibria can be computed, we review the results from extreme value theory applied to the current problem. Case (a) corresponds to those networks where resources are too scarce compared to the demand. This case is similar to the N user case studied in the previous subsection. Case (b) corresponds to networks with better rates of available resources over demand.

Case a. Let $c = c(n) \in \{1, .., n\}, n \in \mathcal{N}$ and $c \to \infty$, $c/n \to 0$. Denote by $p_n = (n - c)/n$ and assume that F_{X_L}, F_{X_H} satisfy the von Mises conditions, i.e., both are extremable functions; see Proposition 7. Therefore, the following result is proven; see [30]:

$$\begin{aligned} &n f_{X_L}(F_{X_L}^{-1}(p_n)) c^{-1/2} (Y_n^c - F_{X_L}^{-1}(p_n)) \xrightarrow{d} N(0,1) \\ &n f_{X_H}(F_{X_H}^{-1}(p_n)) c^{-1/2} (Z_n^c - F_{X_H}^{-1}(p_n)) \xrightarrow{d} N(0,1), \end{aligned} \tag{23}$$

where f_{X_L} and f_{X_H} are the probability density functions of F_{X_L} and F_{X_H}, respectively.

Case b. Let $c = c(n) \in \{1, .., n\}, n \in \mathcal{N}$ and $c \to \infty$, $c/n \to p$. The limit distribution for the intermediate order statistics Y_c^n and Z_c^n is given by Theorem 5.8 in [31], where the authors explore for a fixed $p \in [0, 1]$ the set $D(p)$ of all distribution functions F for which the c order statistic is asymptotically normal for all sequences $c/n \to p$. The following is a restatement of the theorem using the notation in our paper.

Suppose that F_{X_i} is differentiable in y_o, where $p = F_{X_i}(y_o)$ and $f_{X_i}(y_o) > 0$. For any sequence $c(N_i) = pN_i + O(\sqrt(N_i))$, the sequence $Y_n^c(Z_n^c)$ is asymptotically normal $N(\mu_{N_i}, \sigma_{N_i}^2)$ with $\mu_{N_i} = y_o + (c - N_i p)/N_i f_{X_i}(y_o)$ and $\sigma_{N_i}^2 = p(1 - p)/N_i f_{X_i}(y_o)^2$.

This result can be stated as whenever the distribution function, i.e., the bid distribution function, has a derivative at the point $p = c/n$ the intermediate order distribution converges to a normal, so that we can use this asymptote to calculate both probabilities and expected payments.

Using the above formulas and the existence of a Nash equilibrium, for every user $u \in H$ and Q, we have a tuple (q^+ and q°). They are found employing an algorithm that finds the zeros of the following functions:

$$\begin{array}{l}\pi_u^+(s_L, q) - \pi_u^+(s_H, q),\\ \pi_u^\circ(s_L, Q, q) - \pi_u^\circ(s_H, Q, q).\end{array} \tag{24}$$

Now, we derive an approach to calculate a general q° and decide which high-budget users are promoted to auction $\mathcal{L}$ before establishing the final allocations. The reader can argue that giving this power to the system generates another problem, which is the provider does not apply the selection rule and it maintains all high-budget users in $\mathcal{H}$. However, this behavior can be controlled by users. The provider, who is interested in maintaining a low q°, must announce a high Q reward, so this value is a direct signal given to users of their possibility of being untruthful, see section 7.2. In the following, we show how to calculate a system's q^+ and q° given Q.

5.3 Calculating a general q^+ and q°

We employ an ex-ante assumption, where the provider has an estimate for the number of users competing in every auction, but not their actual bids, so that the best guess is to employ the expected equilibrium probability. This resolving path implies finding the zeros of the expected q^+, q° functions:

$$\begin{array}{l}g^+(q) = E[\pi_u^+(s_L, q) - \pi_u^+(s_H, q)],\\ g^\circ(Q, q) = E[\pi_u^\circ(s_L, Q, s) - \pi_u^\circ(s_H, Q, q)].\end{array} \tag{25}$$

We develop the procedure only for $g^+(q)$ because the same procedure can be used for g°. Expanding $g^+(q)$, we have:

$$g^+(q) = E[\pi^+((\mathcal{H}, w), q) - \pi^+((\mathcal{L}, b_L), q)|w < v] + E[\pi^+((\mathcal{H}, w), q) - \pi^+((\mathcal{L}, b_L), q)|w \geq v] \tag{26}$$

with

$$E[\pi^+((\mathcal{H}, w), q)|w < v] = \int_{b_L}^{1} \int_{w<v} \pi^+((\mathcal{H}, w), q) f(v, w) dw dv. \tag{27}$$

Note that the other parts of $g^+(q)$ can be derived following the same steps.

We are then interested in studying the behavior of this expected value when $N_H \to \infty$ and $N_L \to \infty$, i.e., $\lim_{N_H \to \infty} E[\pi^+((\mathcal{H}, w), q)|w < v]$. Applying the Gauss-Hermite formula to the first part of equation 27 and denoting $n_j = n(x_j)$, we have by observing from the definition of n_j that $n_j \to \infty$. By applying the dominated convergence theorem to the following valid inequalities:

$$\begin{array}{l}vF_{Z^c_{n_j}}(w) \leq v\\ \int_{b_L}^{w} y f_{Z^c_{n_j}}(y) \leq F^{-1}_{X_{H_w}}(1) = w, \quad \text{from (21)}\end{array}$$

and by the uniform integrability of $yf_{Z^c_{n_j}}(y)$, which is a consequence of Proposition 6 extended to the c order statistic and Theorem 4.4.9 [28](pp:79), we have:

$$E[\pi^+((\mathcal{H},w),q)|w<v] = \sum_{j=1}^{m} \frac{\alpha_j}{\sqrt{\pi}} \int_{b_L}^{1} \int_{b_L}^{v} \lim_{n_j\to\infty} \left[\frac{F_{Z^k_{n(x_j)}}(w)v - \int_{b_L}^{w} y f_{Z^k_{n(x_j)}}(y)dy}{(1-b_L)^2} \right] dwdv. \tag{28}$$

At this point, we employ the limit distributions for intermediate statistics. Applying Theorem 5.8 in [31] to $E[\pi^+((\mathcal{H},w),q)|w<v]$, we obtain:

$$E[\pi^+((\mathcal{H},w),q)|w<v] = \sum_{j=1}^{m} \frac{\alpha_j}{\sqrt{\pi}} \int_{b_L}^{1} \int_{b_L}^{v} \left[\frac{\Phi(w)v - \int_{-\infty}^{w} y\phi(w)dy}{(1-b_L)^2} \right] dwdv, \tag{29}$$

where $\Phi(w)$ and $\phi(w)$ correspond to the cumulative and density, respectively, of a normal distribution with mean μ_{n_j} and standard deviation σ_{n_j} defined by the Theorem 5.8 in [31]. This last equation is integrable using once again the Hermite formula with limits $-\infty, 0$ and $0, w$.

Regarding the computational complexity, this method is simple because of the use of the Hermite formula. In fact, polynomials accompanying e^{-y^2} have a low degree; thus, we only need few (i.e., 5) roots of the order Hermite polynomials. The roots and weights are pre-calculated and stored for the mechanism's execution.

5.4 Calculating Q

We have approximated $g^+(q)$ and $g^\circ(Q,q)$[5]. We now introduce an optimization problem that generates the Q that minimizes the expected probability of users going into the L auction and simultaneously controls the measure z (given as a parameter). This formulation minimizes the expected probability q° to be used on mechanisms giving that the z_{tr} measure is greater than the threshold ($\overline{z_{tr}}$).

$$\begin{aligned} &\underset{q,s,Q}{\text{minimize}}\ s \\ &\text{subject to} \\ &-g^\circ(Q,s) \le 0, \quad g^+(q) = 0, \quad q - s \ge \overline{z_{tr}}Q \\ &\quad 0 \le q \le 1, \quad 0 \le s \le 1, \quad 0 \le Q \le 1. \end{aligned} \tag{30}$$

Finally, a gradient method can be used to resolve this problem.

[5] This approximation is the same as the one used for the binomial distribution.

6 Testbed Description

In the following, we describe the testbed setup and the configuration used to obtain quantitative results.

6.1 Auction Setup

Operators assign resources for the time interval $[0, T]$. An initial partition $\mathcal{T}$ of $[0, T]$ is formed such that $0 = t_0 \leq\ t_1 \leq \cdots \leq t_S = T$. $[t_s, t_{s+1}), \forall s \in \{1, S\}$ represents a time slot to be auctioned. For every time slot, the link capacity is divided into units of equal size s. Users should become involved in periodic auctions, which are available for future time slots.

Five different auction models are included for testing: (1) a second uniform price auction (SUPA), where users send an ordered list $l = (n, x, q, t_s, t_f)$, where n indicates the traffic key, x is the price per unit, q is the requested unit amount, t_s is the start date time, and t_f is the end date time. (2) A subsidy auction (SA), which takes the ordered list l and assigns subsidies (20%) to those users who bid less than a threshold (b_L). (3) A PSP auction, which also takes the ordered list l. (4) A second uniform price auction with perfect information (PIM), where users decide truthfully the auction to enroll based on their budget. For this case, users add another parameter i for indicating the auction to the ordered list l. And finally, an auction following the assign and price rules proposed by RwP. In this case, users send the same ordered list as l. All simulations were executed in a new auction system that has already been implemented [6]

6.2 User Session Characteristics

Following the conclusions of [24], we model user arrivals following a non-homogeneous Poisson process. Arrival rates per hour are calculated using the data reported in [32] for weekdays. We choose this reference because it reports traffic statistics from a rural network that has the demographic characteristics being addressed by this work. Although the authors suggest this scenario as an extreme case, where there are few users compared to commercial networks. Being aware of this condition and the fact that intermediate order statistics depend on the relationship between capacity and demand, we scale the traffic to study the mechanism's behavior under the double rate, where most of the time slots have less capacity than demand. This scenario is called $2x$. We run tests using 32 units of capacity.

The authors in [33] provide empirical cumulative distribution functions (CDPs) for session length and average data rate per session. Their experiments are built over a commercial Wi-Fi hotspot provider running operations all

[6] An open source implementation for the auction system is available at: `https://github.com/lmarent/Auction_Engine.`

over Australia. Information is presented for three different user categories by contract type. We take the reports for the ALL category. For the session length, we adopt an exponential distribution with the reported mean of 24 minutes. This assumption is not completely arbitrary. In fact, the group tracking the traffic network evolution in [32] has suggested that, based on its observations, traffic is comparable to that in urban settings or the general Internet. The number of units requested in a particular session follows a discrete probability distribution computed from the CDF of the average data rate per session (see Figure 12 in [33]).

7 Results and Observations

In this paper, we study the challenge of dealing with untruthful users for social goal mechanisms. We use the RwP mechanism to support the hypothesis that prices can be set to cover the targeted group, even in periods of high demand; see Subsection 7.1. As part of this study, we first present the results for simulations, which indicate that the RwP scheme reduces 20% of revenue and reaches 100% higher coverage for the targeted group compared with SUPA. Then, we show how the coverage and prices change using different mechanisms and how these measures are modified with the relationship between capacity and demand in scenarios $2x$, $3x$ and $4x$. In Subsection 7.1.5, we present results on subsidized auctions as an alternative to set-aside schemes. Based on them, we suggest that it is better to use the RwP scheme. Finally, Subsection 7.2, analyze how the equilibrium probability q° evolve as important auction variables change.

7.1 Auction Outcomes

The PIM, SUPA and PSP mechanisms are used to benchmark the outcomes in terms of efficiency and revenue. The RwP mechanism fixes the price for the low-budget set and, by this way, it loses income. We track the income variation to determine this cost. Likewise, efficiency is compromised because units that should be sold to high-budget users are allocated to low-budget users. How much it modifies coverage for both groups is, therefore, indicating whether the efficiency cost is worth it.

7.1.1 Results on Network's Income

Surprisingly, PIM not always generates less income than operating with SUPA or PSP; see Table 3. Thus, a provider using this mechanism should establish capacity within a region, so that the PIM mechanism produces more income than operating with other auction mechanisms. The additional income can be interpreted as saying that the second uniform price rule, in the case of SUPA, or the social opportunity cost, in the case of PSP, favors high-budget users, which indirectly take advantage of the poor to achieve lower prices.

However, as we expect, income gains from RwP does not prevail as we increase demand, which is shown in Table 3 for the $2x$ scenario. These results indicate that set-aside schemes create more competition in the high-budget set, which drives their prices up, compensating the operator's income.

	Income						Coverage (%)			
	1x			2x			1x		2x	
	HBS	LBS	Total	HBS	LBS	Total	HBS	LBS	HBS	LBS
RwP	8927	0	8927	28857	0	28857	78	53	57	38
PSP	6718	1375	8093	32438	4177	36615	85	45	63	31
PIM	13436	2237	15673	48965	10667	59632	82	49	58	37
SUPA	6962	1702	8664	42739	7958	50697	85	48	63	31
Subsidy Auction	1647	383	2030	15722	1617	17339	84	45	63	32

Table 3 Network income and coverage for the mechanisms evaluated.

7.1.2 Allocations per Group

Table 3 also presents coverage. The results indicate that RwP achieves the desired behavior. It increases the low-budget group coverage and decreases the high-budget group coverage. For peak periods, unlike SUPA or PSP the coverage gap between the two groups is reduced as we increase demand. Observe that RwP gives even more coverage for low-budget users than PIM as a consequence of how the units are allocated between auctions. RwP assigns dynamically unused units from the high-budget auction to the low-budget auction when the sum of requested units are less than capacity.

7.1.3 Coverage vs. Network Income Trade Off

Figure 3 shows peak interval's income and coverage relative to those in PIM, the ideal mechanism when everybody is truly revealing their information. RwP outperforms all the other auction mechanisms for demand $1x$, not only it reduces the coverage gap, but also it earns more income from the market. Observe that SUPA results are better than those of PSP, which is the consequence of using the social opportunity cost (SOC). The SOC rule produces, in general, lower prices as long as there exist users with low valuations for units, as it is the case for the network scenario being studied in this paper.

RwP outperforms PSP for simulations executed with demand $2x$. In order to show this assertion, we start to decrease the number of units reserved to low-budget auction, so that the RwP allocations approaches the SUPA auction behavior –to have only one auction. The resulting trend is shown in Figure 3, where allocations move from the bottom-right figure area towards $SUPA2x$ upper-left area. This path represents a Pareto front for auction mechanisms. Results are suggesting that there exists a Pareto dominance for some pre-allocation that makes RwP more appropriate than PSP.

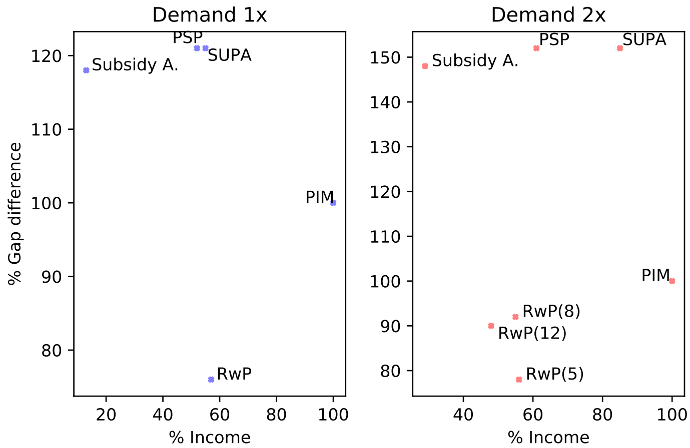

Fig. 3 Income and coverage relationship for peak intervals. For this graph the reference auction is PIM. The ideal mechanism should be the one that decreases gap difference and maintains the greatest amount of income, in other words, low-left figure's area

To sum-up the income-coverage relationship there are important issues in terms of agent optimal actions. For low-budget users, it is clear than RwP increases the coverage as long as c_l increases. From dominance in $1x$, it can be suggested that operators are encouraged to increase the coverage within some region for c_l, *which is supporting the hypothesis that a set-aside scheme can be used to guarantee a minimal spread of resources*. In addition, these results can be used for regulators in order to establish low-budget target coverage by dictating over the rate capacity/demand and c_l, c_h values.

7.1.4 Price Evolution

Figures 4 and 5 present the frequency and average price per hour using mechanisms RwP and SUPA (one auction), respectively. The price for the low-budget set is in the range $0.0 - 0.1$; thus, more than 60% $(1 - F_{Y_L}(0.1))$ of the low-budget population is able to use services. *These results support the hypothesis that it is possible to set affordable prices.* Of equal importance, Figure *4 shows that self price discrimination is possible.* Low budget prices in peaks correspond to low-demand hours, when all users should compete for resources. Even on those intervals, prices are lower than those set by SUPA.

7.1.5 Is It Better to Use a Subsidy Mechanism?

In the selling bandwidth case, these results contradict the authors in [10], where it is suggested that a subsidy auction is better for coverage and revenue combined than a set-aside scheme. In terms of income, subsidy auctions consistently produce a lower income and they do not have any considerable

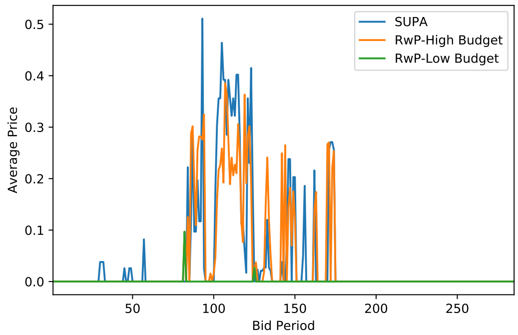

Fig. 4 Average price per bidding slot, mechanisms RwP (high and low-budget) and SUPA (One Auction)

effect on the targeted group coverage, which is almost the same coverage as SUPA. Consequently, we suggest that using a set-aside scheme is the correct alternative for spreading resources.

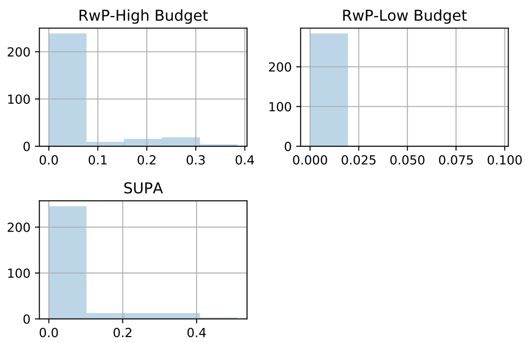

Fig. 5 Price frequency, mechanisms RwP (high and low-budget) and SUPA (One Auction)

7.2 Sensitivity Analysis of q^+ and q°

Table 4 shows the corresponding q^+ and q° for mechanisms as long as Q and N_H increase. As we expect, the equilibrium probabilities (q°) are decreasing in Q and increasing in N_H. This last part can be understood as follows: a high-budget user is willing to accept more competition for the same units whenever a higher reward is offered. Although, the RwP mechanism may establish Q values that makes to be truthful a pure Nash equilibrium, this condition is infrequent and an operator aiming to improve revenues should expect a positive q°. For users, a higher Q value signals less opportunity for the provider not performing the auction selection.

Table 4 Expected probability of using the s_L strategy at the Nash equilibrium under mixed strategies. Parameters: $N_L = 70$, $N_H \in [30, 70]$, $c_H = c_L = 16$, $b_L = 0.5$, and $P_H = 0.5$.

	Two-Auction Mechanism		RwP Mechanism	
Q	Avg q^+	Std desv q^+	Avg q°	Std desv q°
0	0.5438	0.0153	0.2871	0.0583
0.1			0.2669	0.0538
0.2			0.2028	0.0519
0.3			0.1517	0.0564
0.4			0.1036	0.0603
0.5			0.0694	0.0532

We also verify the outcomes of different parameters; such as, quantities assigned on both sets and reserve prices. Greater quantities assigned to the low-budget set as well as an increase in the reserve price of the high budget auction make q° to grow. To preallocate greater quantities for the $\mathcal{H}$ auction than those for $\mathcal{L}$ has the opposite effect.

8 Conclusions

Auctions have been identified as a vehicle to increase efficiency and revenues for selling bandwidth. Studies have also shown that auctions can be used to increase the coverage of resources. In this paper, we propose a set-aside scheme for bandwidth allocation that may enlarge the network access of poor populations, but an untruthful user behavior problem arises. To solve this problem, we design the RwP mechanism, giving rewards to truthful users and reducing the chance of an allocation for untruthful users. We studied the underlying game, showing the existence of a unique Nash equilibrium in mixed strategies and how to calculate it.

We benchmarked RwP against three previously proposed mechanisms: a subsidized auction for the targeted group, a progressive second price auction, and a second price auction for multiple units. The results indicate that (1) using a single auction drives coverage for the targeted group to starvation (which also happens with subsidized auctions) and (2) RwP prevents starvation with a 3% increase in revenues against the second price auction mechanism. Hence, a certain coverage guarantee can be assured. We suggest that RwP successfully addresses the problem of untruthful users. Therefore, these results suggest that our proposal provides an effective operational approach to expand network access and to reduce the global digital divide.

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A Collection of Proofs

Proposition 8 *Assume that users signal their true type. The optimal bidding strategy for user u is to send a bid equal to $\beta_u = min\{v_u, w_u\}$.*

Proof (Proof of Proposition 8) Assume there are N users in the auction and $\{\beta_u, 1 \leq u \leq N\}$ their corresponding bids. For this proof, we refer to the highest bid in the auction among all users different from u as $p_{-u} = max\{\beta_j, 1 \leq j \leq N, j \neq u\}$. Assume first that the user u sends a bid β_u higher than her budget realization w_u. When $p_{-u} < \beta_u$ she wins. If $p_{-u} > w_u$, then user u is not able to pay for the unit and has to pay a fine D, so the net profit is $-D$. If $p_{-u} \leq w_u$, then he/she would have won bidding w_u. Therefore, irrespective of the value, it is not optimal to bid above w_u. If $v_u < w_u$, then it is not optimal to send a value greater than v_u due to in case of winning she may obtain a negative profit (whenever $p_{-u} > v_u$) or the same profit that she would have obtained in case of sending v_u ($p_{-u} \leq v_u$). Finally bidding $\beta_u < \min\{v_u, w_u\}$ is not optimal. The user loses the profit $v_u - p_{-u}$ whenever $\beta_u < p_{-u} \leq v_u$ and receives the same profit if $p_{-u} \leq \beta_u$. Summing up the three results presented, the optimal strategy for any user u is to bid $\beta_u = \min\{v_u, w_u\}$.

Proposition 9 *Assume a user $u \in H'$. If u decides to participate in auction $\mathcal{L}$, then $\beta_u = b_L$.*

Proof (Proof of Proposition 9) Assume that user u has realizations $v_u, w_u > b_L$ and sends a bid $\beta_u < b_L$. Also, assume that $n_L \geq N_L$ other users participate in the auction $\mathcal{L}$. That is N_L low-budget users and $n_L - N_L$ high-budget users. For the same arguments as in proposition 8 the optimal bidding strategy is to place a bid $\min\{v_u, w_u\}$; however, doing that, she is sending to the auctioneer her true type, high budget, so she is assigned to the auction $\mathcal{H}$. As in proposition 8 denote by p_{-u} the highest competing bid for the n_L users. If the user bids b_L and $p_{-u} < b_L$, then the payoff is equal to $v_u - p_{-u}$. The same payoff is obtained bidding $\beta_u < b_L$ when $p_{-u} < \beta_u$, but bidding $\beta_u < b_L$ leads to payoffs equal to zero when $b_L \geq p_{-u} > \beta_u$, then it is optimal for user u once she decides going into $\mathcal{L}$ to bid b_L.

Proof (Proof of Proposition 1) By definition:

$$E[Z^1_{n_H} | Z^1_{n_H} \leq \beta_u] = \int_0^{\beta_u} \frac{y n_H (F_{X_H}(y))^{n_H - 1} f_{X_H}(y) dy}{(F_{X_H}(\beta_u))^{n_H}},$$

integrating by parts with $u = y$ and $dv = n_H (F_{X_H}(y))^{n_H - 1} f_{X_H}(y) dy$, we obtain:

$$E[Z^1_{n_H} | Z^1_{n_H} \leq \beta_u] = \beta_u - \int_0^{\beta_u} \left(\frac{F_{X_H}(y)}{F_{X_H}(\beta_u)} \right)^{n_H} dy.$$

Observe now that $\frac{F_{X_H}(y)}{F_{X_H}(\beta_u)} \leq 1, 0 \leq y \leq \beta_u$, which means for $0 < n_{H_1} < n_{H_2} \leq N_H$:

$$-\int_0^{\beta_u} \left(\frac{F_{X_H}(y)}{F_{X_H}(\beta_h)} \right)^{n_{H_1}} dy \leq -\int_0^{\beta_u} \left(\frac{F_{X_H}(y)}{F_{X_H}(\beta_h)} \right)^{n_{H_2}} dy;$$

therefore:

$$E[Z^1_{n_{H_1}} | Z^1_{n_{H_1}} \leq \beta_u] \leq E[Z^1_{n_{H_2}} | Z^1_{n_{H_2}} \leq \beta_u]$$

Proof (Proof of Proposition 2)
Let $0 \leq k_1 < k_2 \leq N_H$ and $n_1 = N_H - k_1, n_2 = N_H - k_2$. It is required to show that $\mu_u^+(s_H, n_2) \geq \mu_u^+(s_H, n_1)$. From $k_2 > k_1 \Rightarrow n_2 < n_1$. By Proposition 1

$$\mu_u^+(s_H, n_2) \geq \mu_u^+(s_H, n_1),$$

therefore, applying Proposition 11, $\pi_u^+(s_H, q)$ is a non-decreasing function of q. For the second part, we need to prove that $\mu_u^+(s_L, n_1) \geq \mu_u^+(s_L, n_2)$

$$\begin{aligned}\mu_u^+(s_L, n_1) &= \frac{v_u - E[Y_{N_L}^1]}{N_H+1-N_H+k_1} \\ &> \frac{v_u - E[Y_{N_L}^1]}{k_2+1} \\ &= \mu_u^+(s_L, n_2);\end{aligned}$$

therefore, by Proposition 10, $\pi_u^+(s_L, q)$ is a non-increasing function of q.

Proposition 10 *Let $f(n)$ be a non-increasing function in n; $f(n) \geq 0, 0 \leq n \leq N$. Then,*

$$\pi(q, k) = \sum_{j=0}^{k} B(j, N, q) f(j) \tag{31}$$

is a non-increasing function of q.

Proof (Proof of Proposition 10)
Let $G(q, k) = \sum_{j=0}^{k} B(j, N, q)$ and $0 \leq q_1 < q_2 \leq 1$. First, we show that $G(q_1, k) \geq G(q_2, k), \forall k \in [0, N]$. Observe that there exists a $k^\diamond$ such that:

$$\begin{aligned}B(k, N, q_1) \geq B(k, N, q_2), \forall k \in [0, k^\diamond] \\ B(k, N, q_1) \leq B(k, N, q_2), \forall k \in (k^\diamond, N]\,;\end{aligned} \tag{32}$$

therefore, $\forall k \in [0, k^\diamond]$,

$$\begin{aligned}G(q_1, k) &= \textstyle\sum_{j=0}^{k} B(k, N, q_1) \\ &\geq \textstyle\sum_{j=0}^{k} B(k, N, q_2) \\ &= G(k, q_2).\end{aligned} \tag{33}$$

Moreover, for $k \in (k^\diamond, N]$, we have:

$$G(q_1, k) = 1 - \sum_{j=k+1}^{N} B(k, N, q_1), \tag{34}$$

and since

$$\sum_{j=k+1}^{N} B(k, N, q_1) \leq \sum_{j=k+1}^{N} B(k, N, q_2),$$

then $G(q_1, k) \geq G(q_2, k)$.

Now, we proceed to prove the result by induction. The base case corresponds to $k = 0$; that is:

$$\begin{aligned}\pi(q_1, 0) &= B(0, N, q_1) f(0) \\ &= G(q_1, 0) f(0) \\ &> G(q_2, 0) f(0) \\ &= B(0, N, q_2) f(0) \\ &= \pi(q_2, 0).\end{aligned} \tag{35}$$

Assume that $\pi(q_1, k) \geq \pi(q_2, k)$; the partial sum up to $k + 1$ is equal to:

$$\pi(q_1, k+1) = \bar{a} G(q_1, k) + B(k+1, N, q_1) f(k+1) \tag{36}$$

$$\pi(q_2, k+1) = \bar{b} G(q_2, k) + B(k+1, N, q_2) f(k+1) \tag{37}$$

with $\bar{a} = \frac{\pi(q_1,k)}{G(q_1,k)}$ and $\bar{b} = \frac{\pi(q_2,k)}{G(q_2,k)}$.

Case 1. $k+1 \leq k^\diamond$. Then, we have that $B(k+1, N, q_1) f(k+1) \geq B(k+1, N, q_2) f(k+1)$. Thus, we sum both inequalities and obtain $\pi(q_1, k+1) \geq \pi(q_2, k+1)$.

Case 2. $k + 1 > k^\diamond$. Then, we have the following two inequalities:

$$B(k+1, N, q_1) f(k+1) \leq B(k+1, N, q_2) f(k+1) \tag{38}$$

$$\begin{array}{ll} G(q_1,k+1) & \geq G(q_2,k+1) \\ G(q_1,k)+B(q_1,N,k+1) & \geq G(q_2,k)+B(q_2,N,k+1) \\ G(q_1,k)-G(q_2,k) & \geq B(q_2,N,k+1)-B(q_1,N,k+1). \end{array} \tag{39}$$

Observe that $\overline{a} \geq \overline{b} \geq f(k+1)$ because $f(n)$ is non-increasing; thus, we have the following inequality:

$$\overline{a}G(q_1,k)-\overline{b}G(q_2,k) \geq \overline{b}(G(q_1,k)-G(q_2,k)), \tag{40}$$

adding $f(k+1)(B(q_2,N,K)-B(q_1,N,K))$ to both sides and using 39, we obtain:

$$\pi(q_1,k+1)-\pi(q_2,k+1) \geq 0 \tag{41}$$

which completes the proof.

Proposition 11 *Let $f(n)$ be a non-decreasing function in n; $f(n) \geq 0, 0 \leq n \leq N$. Then,*

$$\pi(q,N)=\sum_{j=0}^{N} B(j,N,q)f(j) \tag{42}$$

is a non-decreasing function of q.

Proof (Proof of Proposition 11)
Define $G^{-1}(a,q):[0,1] \longrightarrow [0,N]$, with $G(k,q)$ defined as in Proposition 10

$$G^{-1}(a,q)=sup\{k \geq 0 : G(k,q) \leq a\}. \tag{43}$$

The reader can verify that:

$$\pi(q,N)=\int_0^1 G^{-1}(a,q)f(G^{-1}(a,q))da. \tag{44}$$

Observe that $G(q_1,k) \geq G(q_2,k) \Rightarrow G^{-1}(a,q_1) \leq G^{-1}(a,q_2)$. Moreover, by assumption, $f(n)$ is non-decreasing; therefore, $f(G^{-1}(a,q_1)) \leq f(G^{-1}(a,q_2)), \quad \forall a$. Thus, $\forall a \in [0,1]$,

$$G^{-1}(a,q_1)f(G^{-1}(a,q_1)) \leq G^{-1}(a,q_2)f(G^{-1}(a,q_2)); \tag{45}$$

therefore, by the monotonicity of the Lebesgue integral:

$$\begin{array}{ll} \pi(q_1,N) & =\int_0^1 G^{-1}(a,q_1)f(G^{-1}(a,q_1))da \\ & \leq \int_0^1 G^{-1}(a,q_2)f(G^{-1}(a,q_2))da \\ & =\pi(q_2,N). \end{array} \tag{46}$$

Proof (Proof of Proposition 3) By definition

$$\begin{array}{ll} \pi_u(s_H,q) & =\sum_{k=0}^{N_H-1} B(k,N_H,q)\mu_u(s_H,N_H-k)+q^{N_H}(v_u-p_H) \\ & \geq \sum_{k=0}^{N_H-1} B(k,N_H,q)\mu_u(s_H,N_H)+q^{N_H}(v_u-p_H), \quad \text{because of 7} \\ & =\mu_u(s_H,N_H)(1-q^{N_H})+q^{N_H}(v_u-p_H). \end{array}$$

The last term is continuous and increasing in q. It starts at $\mu_h(s_H,N_H)$ and has a maximum in $q=1$, where the function is equal to v_u-p_H. The expected payoff for strategy s_L is continuous and decreasing in q. Also, observe that $\pi_u(s_L,q)$ starts at $v_u-E[Y^1_{N_L}]$ and ends at $\frac{v_u-E[Y^1_{N_L}]}{N_H+1}$. From the condition $\frac{N_H v_u+E[Y^1_{N_L}]}{N_H+1} \geq p_H, \quad \forall u \in H$ and $v_u-E[Y^1_{N_L}] \geq \mu_u(s_H,N_H)$, we have $\forall u \in H$: $\pi_u(s_L,q)-\mu_u(s_H,N_H)(1-q^{N_H})+q^{N_H}(v_u-p_H)$, must have a zero (q^*) in the compact set $[0,1]$.

Moreover, by definition $\pi_u(s_H,0)=\mu_u(s_H,N_H)$ and $\pi_h(s_H,1)=v_u-p_H$ and continuity of $\pi_u(s_H,q)$, it must exist at least a $q^+ \leq q^*$, such that:

$$\pi_u(s_H,q^+)=\pi_u(s_L,q^+), \tag{47}$$

which completes the proof.

Proof (Proof of Proposition 6)

Apply the second derivative criterion for all functions and verify that it is positive; for instance, we have for $\frac{1}{F_{X_L}}$:

$$\frac{d^2}{dy^2}\left(\frac{1}{F_{X_L}(y)}\right) = \frac{2b_L(b_L+1-2y)^2}{(yb_L+y-y^2)^3} + \frac{2b_L}{(yb_L+y-y^2)^2} > 0 \tag{48}$$

Proof (Proof of Proposition 7)

To prove that F_{X_L} belongs to the Weibull family [28](pp:299), it is necessary to prove for $\xi_1 = F_{X_L}^{-1}(1)$ and F_{X_L} that:

$$\lim_{y\to b_L}(b_L-y)\left(\frac{b_L+1-2y}{b_L}\right)\left(1-\left(\frac{yb_L+y-y^2}{b_L}\right)\right)^{-1} = \alpha,$$

applying L'Hopital's rule to the resulting undetermined form, we have:

$$\lim_{y\to b_L}\frac{(-3b_L-1+4y)}{(-b_L+2y-1)} = 1 = \alpha$$

and for F_{X_H}, we verify, applying L'Hopital's rule two times, that:

$$\lim_{y\to 1}\frac{(1-y)(2-2y)}{(1-b_L)^2-\left(2y-y^2-2b_L+b_L^2\right)} = \lim_{y\to 1}\frac{2-4y+2y^2}{1-2y+y^2} = 2 = \alpha$$

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