Projected Dynamical Systems and Applications

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Projected Dynamical System (PDS)

$$\frac{dx(t)}{dt} = \Pi_{\boldsymbol{K}}\left(x(t), -F\left(x(t)\right)\right) \qquad (PDS)$$

$\boldsymbol{K} \subseteq \boldsymbol{R}^n$: convex polyhedral set;

$F : \boldsymbol{K} \to \boldsymbol{R}^n$: Lipschitz continuous function with linear growth;

$\Pi_{\boldsymbol{K}} : \boldsymbol{R} \times \boldsymbol{K} \to \boldsymbol{R}^n$: Gateaux directional derivative

$$\Pi_{\boldsymbol{K}}\left(x, -F(x)\right) = \lim_{\delta \to 0^+} \frac{P_{\boldsymbol{K}}\left(x - \delta F(x)\right) - x}{\delta};$$

$P_{\boldsymbol{K}} : \boldsymbol{R}^n \to \boldsymbol{K}$: projection operator $\left\| P_{\boldsymbol{K}}(z) - z \right\| = \inf_{y \in \boldsymbol{K}} \left\| y - z \right\|$;

Theor. [Dupuis-Nagurney (1993)]

Critical points of (PDS) $\Leftrightarrow$ solutions to

$$\text{Find } x \in \boldsymbol{K} : \langle F(x), y - x \rangle \geq 0, \quad \forall\, y \in \boldsymbol{K}.$$

Isac-Cojocaru (2002, 2004): study of PDS in infinite-dimensional Hilbert spaces

Lions-Stampacchia (1967), Brezis (1967): introduction of Evolutionary Variational Inequalities (EVI)

Steinbach (1998): study of obstacle problems by means of VI

Daniele-Maugeri-Oettli (1998): applications of EVI to network problems

Cojocaru-Daniele-Nagurney (2004): connection between PDS and EVI and introduction of Double-Layered Dynamics

General Formulation of the set K *for traffic network problems, spatial price equilibrium problems, financial equilibrium problems*

$$\mathbf{K} = \bigcup_{t \in [0,T]} \Big\{ u \in L^2\big([0,T], \mathbf{R}^q\big) : \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i = 1,\ldots,q, j = 1,\ldots,l \Big\}$$

$$\lambda, \mu, \rho \in L^p\big([0,T], \boldsymbol{R}^q\big): \text{convex functions}$$

Preliminary Definitions

- $M \subset H \Rightarrow M^0 = \left\{ \xi \in H : \langle \xi, x \rangle \le 1, \ \forall x \in M \right\}$

- C cone $\Rightarrow C^0 = C^- = \left\{ \xi \in H : \langle \xi, x \rangle \le 0, \ \forall x \in C \right\}$

- $\boldsymbol{K} \subset H$ nonempty, closed, convex $\Rightarrow$

  $T_{\boldsymbol{K}}(x) = \overline{\bigcup_{\lambda > 0} \lambda \left(\boldsymbol{K} - x \right)} =$ support cone

- $N_{\boldsymbol{K}}(x) = \left\{ \xi \in H : \langle \xi, z - x \rangle \le 0, \ \forall z \in \boldsymbol{K} \right\} =$

  normal cone to $\boldsymbol{K}$ at x

- **Prop. 1:** $\left(T_K(x)\right)^0 = N_K(x) = \left(T_K(x)\right)^-$

- **Theor. 1:** $P_K\left(x+\lambda h\right) = x + \lambda P_{T_K(x)} h + o(\lambda)$

$$\forall x, h, \lambda > 0$$

- **Cor. 1:** If $\Pi_K\left(x,h\right) = \lim_{\lambda \to 0^+} \frac{P_K\left(x+\lambda h\right) - x}{\lambda}$,

then $\Pi_K\left(x,h\right) = P_{T_K(x)} h.$

- $n_{\boldsymbol{K}}(x)=\{v:\|v\|=1 \text{ and } \langle v, x-y\rangle \leq 0,\ \forall\, y \in \boldsymbol{K}\}=$ set of unit inward normals to $\boldsymbol{K}$ at x

- **Prop. 2:** $n_{\boldsymbol{K}}(x)=\partial B(0,1) \cap -\left(T_{\boldsymbol{K}}(x)\right)^{0}$

- $\operatorname{qi} \boldsymbol{K}=\{x \in \boldsymbol{K}: T_{\boldsymbol{K}}(x)=H\}$

- $\operatorname{qbdry} \boldsymbol{K}=\boldsymbol{K} \setminus \operatorname{qi} \boldsymbol{K}$

- **Prop. 3:** $x \in \operatorname{qbdry} \mathbf{K} \Leftrightarrow n_{\mathbf{K}}(x) \neq \emptyset$

- **Theor. 2:**

$$x \in \mathrm{qi}\,\mathbf{K} \Rightarrow \Pi_{\mathbf{K}}(x,h) = h,\ \forall\, h \in H;$$

$$x \in \mathrm{qbdry}\,\mathbf{K} \Rightarrow \forall\, v \in H \setminus T_{\mathbf{K}}(x), \exists\, n^{*}(x) \in n_{\mathbf{K}}(x):$$

$$\beta(x) = -\left\langle v, n^{*}(x) \right\rangle > 0,$$

$$\Pi_{\mathbf{K}}\left(x,v\right) = v + \beta(x)\, n^{*}(x).$$

- **Cor. 2:**

$$\Pi_{\mathbf{K}}\left(x,v\right) = P_{v-N_{\mathbf{K}}(x)}\left(0\right) = \left(v - N_{\mathbf{K}}(x)\right)^{\#} \quad \forall\, v \in H.$$

$$\text{Cor. 2 } \Rightarrow \exists\,!\, n_x : \Pi_{\mathbf{K}}\left(x, -F\left(x\right)\right) = -F\left(x\right) - n_x$$

where $n_x = 0$ if $x \in \mathrm{qi}\,\mathbf{K}$

➢ Remark 1: Cor. 2 implies

$$\frac{d\dot{x}(t)}{dt} = \Pi_{\boldsymbol{K}}(x, -F(x)) = P_{-F(x)-N_{\boldsymbol{K}}(x)}(0) =$$

$$\left\{ \bar{v} \in -(F(x) + N_{\boldsymbol{K}}(x)) : \|\bar{v}\| = \min_{y \in -(F(x)+N_{\boldsymbol{K}}(x))} \|y\| \right\}$$

$$\begin{cases} \dfrac{d\dot{x}(t)}{dt} = \Pi_{\boldsymbol{K}}(x, -F(x)) \\ x(0) = x_0 \in \boldsymbol{K} \end{cases} \Leftrightarrow$$

slow solution to

$$\dot{x}(t) \in -(N_{\boldsymbol{K}}(x(t)) + F(x(t)))$$

$$x(0) = x_0$$

$$\Pi_{\boldsymbol{K}}\left(x(t),-F\left(x(t)\right)\right)=P_{T_{\boldsymbol{K}}\left(x(t)\right)}\left(-F\left(x(t)\right)\right)$$

$$\Downarrow$$

$$\begin{cases} \dfrac{d\,\dot{x}(t)}{d\,t}=\Pi_{\boldsymbol{K}}\left(x,-F(x)\right) \\ x(0)=x_0\in \boldsymbol{K} \end{cases} \Leftrightarrow$$

slow solution to

$$\begin{cases} \dot{x}(t)\in P_{T_{\boldsymbol{K}}(x)}\left(-F\left(x(t)\right)\right) \\ x(0)=x_0 \end{cases}$$

➢ Remark 2:

$$\text{Find } u \in \mathbf{K}: \ \left\langle\left\langle F(u), v-u \right\rangle\right\rangle \geq 0, \ \ \forall v \in \mathbf{K}$$

$$\text{where} \left\langle\left\langle \Phi, u \right\rangle\right\rangle = \int_0^T \left\langle \Phi(t), u(t) \right\rangle dt,$$

$$\Phi \in L^2\left(\left[0,T\right], \mathbf{R}^q\right)^*, \ \ u \in L^2\left(\left[0,T\right], \mathbf{R}^q\right)$$

$$\Updownarrow$$

$$\text{Find } u \in \mathbf{K}: \ \left\langle F\left(u(t)\right), v(t) - u(t) \right\rangle \geq 0,$$

$$\forall v \in \mathbf{K}, \text{a.e. in } \left[0,T\right]$$

Computational Procedure

$$F \text{ strictly monotone } \Rightarrow$$

$$u(t) \in C^0\left([0,\mathrm{T}], \mathbf{R}^q\right) \text{ unique solution to EVI:}$$

$$\left\langle F\left(u(t)\right), v(t) - u(t)\right\rangle \geq 0, \ \forall t \in [0,T]$$

Partitions of $[0,T]$:

$$\pi_n = \left(t_n^0, t_n^1, \ldots, t_n^{N_n}\right), 0 = t_n^0 < t_n^1 < \ldots < t_n^{N_n} = T$$

$$k_n = \max\left\{t_n^j - t_n^{j-1} : j = 1, \ldots, N_n\right\}, \lim_n k_n = 0.$$

Finite-dimensional Variational Inequality:

$$\left\langle F\left(u\left(t_n^{j-1}\right)\right), v-u\left(t_n^{j-1}\right)\right\rangle \geq 0,\ \forall v \in \mathbf{K}\left(t_n^{j-1}\right) \qquad (FDVI)$$

where

$$\mathbf{K}\left(t_n^{j-1}\right)=\left\{v\in\mathbf{R}^q : \lambda\left(t_n^{j-1}\right)\leq v\leq \mu\left(t_n^{j-1}\right), \sum_{i=1}^{q}\xi_{ji}v_i=\rho_j\left(t_n^{j-1}\right)\right\}$$

Unique solution to $(FDVI) \Leftrightarrow$ critical point of

$$\Pi_{\mathbf{K}}\left(u\left(t_n^{j-1},\tau\right),-F\left(u\left(t_n^{j-1},\tau\right)\right)\right)=0$$

Interpolation function $u_n(t)$:

$$\lim_n \left\|u_n(t)-u(t)\right\|_{L^\infty\left([0,T],\mathbf{R}^q\right)}=0$$

$$P_n\left([0,T],\mathbf{R}^m\right)=\left\{v\in L^\infty\left([0,T],\mathbf{R}^m\right): v_{\left(t_n^{j-1},t_n^j\right]}=v_j\in\mathbf{R}^m,\ j=1,\dots,N_n\right\}$$

Mean value operators $\mu_n : L^1\left([0,T],\mathbf{R}^m\right)\to P_n\left([0,T],\mathbf{R}^m\right)$

$$\mu_n v_{\left(t_n^{j-1},t_n^j\right]}=\frac{1}{t_n^j-t_n^{j-1}}\int_{t_n^{j-1}}^{t_n^j} v(s)\,ds$$

$$\mathbf{K}=\left\{F(t)\in L^2\left([0,T],\mathbf{R}^m\right):\lambda\le F(t)\le\nu,\ \text{a.e. in } [0,T],\right.$$
$$\left.\Phi F(t)=\rho(t),\ \lambda,\nu\ge 0\right\}$$

$C:[0,T]\times\mathbf{K}\to L^2\left([0,T],\mathbf{R}^m\right)$ linear mapping

$$C\left[t,F(t)\right]=A(t)F(t)+B(t),\ \ A(t)\in L^\infty, B(t)\in L^2$$

$$\text{Find } H(t) \in \boldsymbol{K}:$$

$$\int_0^T \left\langle A(t) H(t) + B(t), F(t) - H(t) \right\rangle dt =$$

$$\sum_{j=1}^{N_n} \int_{t_n^{j-1}}^{t_n^j} \left\langle A(t) H(t) + B(t), F(t) - H(t) \right\rangle dt \geq 0, \ \forall F(t) \in \boldsymbol{K}$$

$$\text{Find } u_j^n(t) \in \boldsymbol{K}:$$

$$\int_{t_n^{j-1}}^{t_n^j} \left\langle A(t) H_j^n(t) + B(t), F_j^n(t) - H_j^n(t) \right\rangle dt \geq 0,$$

$$\forall F_j^n(t) \in \boldsymbol{K}, \forall t \in \left[t_n^{j-1}, t_n^j \right]$$

Finite-dimensional problem:

Find $H_j^n \in \boldsymbol{K}_m \subset \boldsymbol{R}^m$:

$$\left\langle A_j^n H_j^n + B_j^n, F_j^n - H_j^n \right\rangle \geq 0, \ \forall F_j^n \in \boldsymbol{K}_m$$

where

$$A_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} A(t)\,dt; \quad B_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} B(t)\,dt.$$

$$H_n(t)=\sum_{j=1}^{N_n}\chi\left(t_n^{j-1},t_n^j\right)H_j^n \ \forall n\in \boldsymbol{N}$$

piecewise constant approximations to solutions to

$$\int_0^T\left\langle A(t)H(t)+B(t),F(t)-H(t)\right\rangle dt\geq 0,\ \forall F(t)\in \boldsymbol{K}?\ (a)$$

- **Theor. 3:**

$A(t)$ positive definite a.e. in $[0,T]\Rightarrow U=\{H_n\}_{n\in N}$ weakly compact; its cluster points are feasible.

If $\overline{H}$ weak cluster point for $U\Rightarrow \overline{H}$ solution to (a)

Time-dependent convex set:

$$\mathbf{K}=\left\{F(t)\in L^2\left([0,T],\mathbf{R}^m\right):\lambda(t)\le F(t)\le \nu(t),\text{a.e. in}[0,T],\right.$$
$$\left.\lambda(t),\nu(t)\ge 0,\Phi F(t)=\rho(t),\text{a.e. in}[0,T]\right\}$$

$$\mathbf{K}_j^n=\left\{F(t)\in L^2\left([0,T],\mathbf{R}^m\right),\text{piecewise constant}:\right.$$
$$\overline{\lambda}_{j,n}\le F_j(t)\le \overline{\nu}_{j,n},\ \text{a.e. in}\left(t_{j-1},t_j\right),$$
$$\left.\Phi F(t)=\overline{\rho}_{j,n},\text{a.e. in}\left(t_{j-1},t_j\right)\right\}$$

where

$$\overline{\lambda}_{j,n}=\mu_{j,n}\,\lambda(t),\ \overline{\nu}_{j,n}=\mu_{j,n}\,\nu(t),\ \overline{\rho}_{j,n}=\mu_{j,n}\,\rho(t)$$

Lemma 1:

The set sequence $\boldsymbol{K}^n = \bigcap \boldsymbol{K}_j^n$ converges to $\boldsymbol{K}$

Theor. 4:

$A(t)$ positive definite a.e. in $[0,T] \Rightarrow$ the sequence

$H^n(t) = \sum_{j=1}^{N_n} \chi\left(t_n^{j-1}, t_n^j\right) H_j^n$ admits weak cluster points.

Each weak cluster point is feasible and solves (a).

Application to Dynamic Traffic Network

Cost functions:

$$C_1\left(H(t)\right) = H_1(t) + 1$$

$$C_2\left(H(t)\right) = H_2(t) + 2$$

$$\boldsymbol{K} = \bigcup_{t\in[0,2]} \left\{ F(t) \in L^2\left([0,2], \boldsymbol{R}^2\right) : 0 \le F_1(t) \le t, 0 \le F_2(t) \le \frac{3}{2} t \text{ a.e.in } [0,2]; \right.$$

$$\left. F_1(t) + F_2(t) = t \text{ a.e. in } [0,2] \right\}$$

Vector field:

$$F: L^2\left([0,2],\mathbf{R}^2\right)\to L^2\left([0,2],\mathbf{R}^2\right)$$

$$\left(F_1\left(H(t)\right),F_2\left(H(t)\right)\right)=\left(H_1(t)+1,H_2(t)+2\right).$$

$t_0\in\left\{\frac{k}{4}:k=0,\ldots,8\right\}\Rightarrow$ sequence of PDS defined by:

$-F\left(H_1(t_0),H_2(t_0)\right)=\left(-H_1(t)-1,-H_2(t)-2\right)$ on

$$\mathbf{K}_{t_0}=\left\{\left\{[0,t_0]\times\left[0,\frac{3}{2}t_0\right]\right\}\cap\left\{x+y=t_0\right\}\right\}$$

Unique equilibrium at t_0:

$$\left(H_1(t_0), H_2(t_0)\right) \in \mathbf{R}^2 : -F\left(H_1(t_0), H_2(t_0)\right) \in N_{\mathbf{K}_{t_0}}\left(H_1(t_0), H_2(t_0)\right)$$

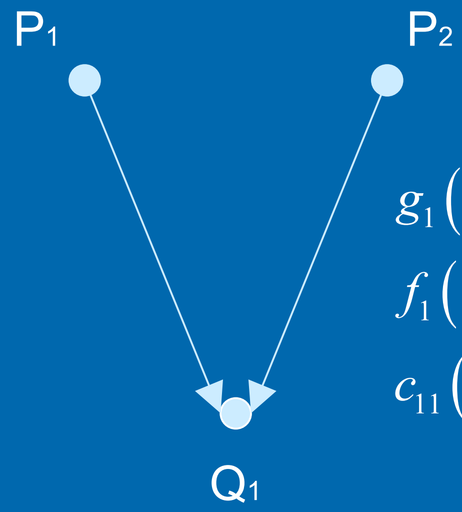
Equilibria:

$$\left\{(0,0), \left(\frac{1}{4}, 0\right), \left(\frac{1}{2}, 0\right), \left(\frac{3}{4}, 0\right), (1,0), \left(\frac{9}{8}, \frac{1}{8}\right), \left(\frac{5}{4}, \frac{1}{4}\right), \left(\frac{11}{8}, \frac{3}{8}\right), \left(\frac{3}{2}, \frac{1}{2}\right)\right\}$$

Explicit formulae:

$$\begin{cases} H_1(t) = t \\ H_2(t) = 0 \end{cases} \quad \text{if } 0 \le t \le 1; \qquad \begin{cases} H_1(t) = \dfrac{t+1}{2} \\ H_2(t) = \dfrac{t-1}{2} \end{cases} \quad \text{if } 1 \le t \le 2.$$

➢ Application to Spatial Price Economic Markets



$$g_1(p(t)) = p_1(t) + h(t); g_2(p(t)) = p_2(t) + k(t);$$

$$f_1(q(t)) = k(t) - q_1(t);$$

$$c_{11}(x(t)) = x_{11}(t); c_{21}(x(t)) = x_{21}(t) + 1;$$

$$\boldsymbol{K} = \left\{ u(t) = (p(t), q(t), x(t)) \in L^2\left(\left[0, \frac{1}{2}\right], \boldsymbol{R}^2\right) \times L^2\left(\left[0, \frac{1}{2}\right], \boldsymbol{R}\right) \times L^2\left(\left[0, \frac{1}{2}\right], \boldsymbol{R}^2\right) : \right.$$

$$0 \le p_1(t) \le h(t) + k(t) + 1, 0 \le p_2(t) \le h(t) + k(t) + 1, 0 \le q_1(t) \le h(t) + k(t) + 1,$$

$$\left. 0 \le x_{11}(t) \le h(t) + k(t) + 1, 0 \le x_{21}(t) \le h(t) + k(t) + 1 \right\}$$

➢ Direct method

$$\begin{cases} v(u(t))=0 \\ u(t)\in \mathbf{K} \end{cases} \Rightarrow \begin{cases} p_1(t)-x_{11}(t)+h(t)=0 \\ q_1(t)+x_{11}(t)+x_{21}(t)-k(t)=0 \\ p_1(t)+x_{11}(t)-q_1(t)=0 \\ x_{21}(t)-q_1(t)+1=0 \\ u(t)\in \mathbf{K} \end{cases} \Rightarrow \varnothing$$

$$\mathbf{K}\cap\{p_2(t)=0\}: \begin{cases} -3h(t)+k(t)+1\geq 0 \\ -h(t)+2k(t)-3\geq 0 \end{cases}$$

$$p_1(t)=\frac{-3h(t)+k(t)+1}{5};\ p_2(t)=0; q_1(t)=\frac{-h(t)+2k(t)+2}{5};$$

$$x_{11}(t)=\frac{2h(t)+k(t)+1}{5}; x_{21}(t)=\frac{-h(t)+2k(t)-3}{5};$$

$$g_1(p(t))=\frac{2h(t)+k(t)+1}{5}; g_2(p(t))=k(t); f_1(q(t))=\frac{h(t)+3k(t)-2}{5};$$

$$c_{11}(x(t))=\frac{2h(t)+k(t)+1}{5}; c_{21}(x(t))=\frac{-h(t)+2k(t)+2}{5}; s_2(t)=\frac{h(t)+3k(t)+3}{5}.$$

- Discretization method: $h(t)=\frac{t}{2},\ k(t)=t+\frac{8}{7}$

$$\int_0^{\frac{1}{2}}\left\langle A(t)u^*(t)+b(t),u(t)-u^*(t)\right\rangle dt\geq 0,\quad \forall u(t)\in \mathbf{K}$$

$$A(t)=\begin{bmatrix}1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 1 & 1 & 1\\ 1 & 0 & -1 & 1 & 0\\ 0 & 1 & -1 & 0 & 1\end{bmatrix},\quad b(t)=\begin{bmatrix}h(t)\\ k(t)\\ -k(t)\\ 0\\ 1\end{bmatrix}$$

$$\left[0,\frac{1}{2}\right]:0<\frac{1}{2n}<\ldots<\frac{j-1}{2n}<\frac{j}{2n}<\ldots<\frac{1}{2}$$

$$\mathbf{K}_j^n = \Big\{ u_j^n(t) = \big(p_{j1}^n(t), p_{j2}^n(t), q_{j1}^n(t), x_{j11}^n(t), x_{j21}^n(t) \big) \text{ piecewise constant:}$$

$$0 \le p_{j1}^n(t) \le \frac{6j-3}{8n} + \frac{15}{7}, 0 \le p_{j2}^n(t) \le \frac{6j-3}{8n} + \frac{15}{7}, 0 \le q_{j1}^n(t) \le \frac{6j-3}{8n} + \frac{15}{7},$$

$$0 \le x_{j11}^n(t) \le \frac{6j-3}{8n} + \frac{15}{7}, 0 \le x_{j21}^n(t) \le \frac{6j-3}{8n} + \frac{15}{7} \Big\}$$

Variational Inequality:

$$\left\langle A(t)\left(u_j^n(t)\right)^* + b_j^n(t), u_j^n(t) - \left(u_j^n(t)\right)^* \right\rangle \ge 0 \quad \forall u_j^n(t) \in \mathbf{K}_j^n(t)$$

$$\text{where } b_j^n(t) = \begin{bmatrix} \dfrac{2j-1}{8n} \\ \dfrac{2j-1}{4n} + \dfrac{8}{7} \\ -\dfrac{2j-1}{4n} - \dfrac{8}{7} \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} A(t)u_j^n(t)+b_j^n(t)=0 \\ u_j^n(t)\in \mathbf{K}_j^n \end{cases} \Rightarrow \varnothing$$

$$\mathbf{K}_j^n \cap \{p_{j2}^n(t)=0\} : \begin{cases} A(t)u_j^n(t)+b_j^n(t)=0 \\ p_{j2}^n(t)=0 \end{cases} \Rightarrow$$

$$\begin{cases} p_{j1}^n(t)=\frac{-2j+1}{40n}+\frac{3}{7};\ \ p_{j2}^n(t)=0;\ \ q_{j1}^n(t)=\frac{6j-3}{40n}+\frac{6}{7}; \\ x_{j11}^n(t)=\frac{2j-1}{10n}+\frac{3}{7};\quad x_{j21}^n(t)=\frac{6j-3}{40n}-\frac{1}{7}; \\ g_{j1}^n(p(t))=\frac{2j-1}{10n}+\frac{3}{7};\ g_{j2}^n(p(t))=\frac{2j-1}{4n}+\frac{8}{7};\ \ f_{j1}^n(q(t))=\frac{14j-7}{40n}+\frac{2}{7}; \\ c_{j11}^n(x(t))=\frac{2j-1}{10n}+\frac{3}{7};\ \ c_{j21}^n(x(t))=\frac{6j-3}{40n}+\frac{6}{7}; \end{cases}$$

Excess: $s_{j2}^n(t)=\frac{14j-7}{40n}+\frac{6}{7};$

Solution on $\mathbf{K}^{n} = \cap \mathbf{K}_{j}^{n}$:

$$u_n(t) = \sum_{j=1}^{n} \chi\left(\frac{j-1}{2n}, \frac{j}{2n}\right) u_j^n(t).$$