

Double-layered dynamics and transportation

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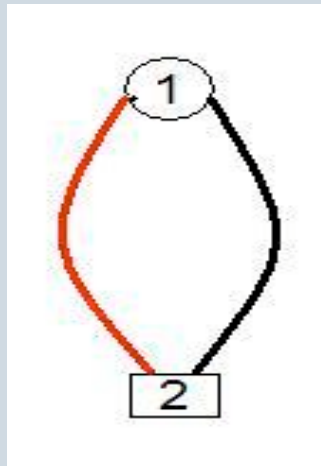
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Outline

- Evolutionary variational inequalities (EVI) → Projected dynamical systems (PDS)
- Double-layered dynamics (DLD)
- Interpretation of time scales
- Discontinuous DLD - Persistence and adjustment time
- Numerical Example
- Final remarks & References

Equilibrium problems

- Problems in economics, operations research, mathematical physics, transportation
- Transportation: 2 locations and 2 links, each link with an associated cost and a flow u_i $i \in \{1,2\}$.



Question #1

Can we model the time evolution of this problem towards an equilibrium state, respecting the constraints $u_j \geq 0$?

Yes, with finite-dimensional VI+PDS theory.
[Nagurney & al.]

Variational Inequalities

- 1964 part of calculus of variations; are used to show existence of equilibria in economic problems & free boundary problems, among others.
- Given a Hilbert space X , K closed, convex in X , $f : K \rightarrow X$, the variational inequality defined by f and K is:
find $x^* \in K$ s.t. $\langle f(x^*), y - x^* \rangle \geq 0, \forall y \in K$.

Projected dynamical systems-I

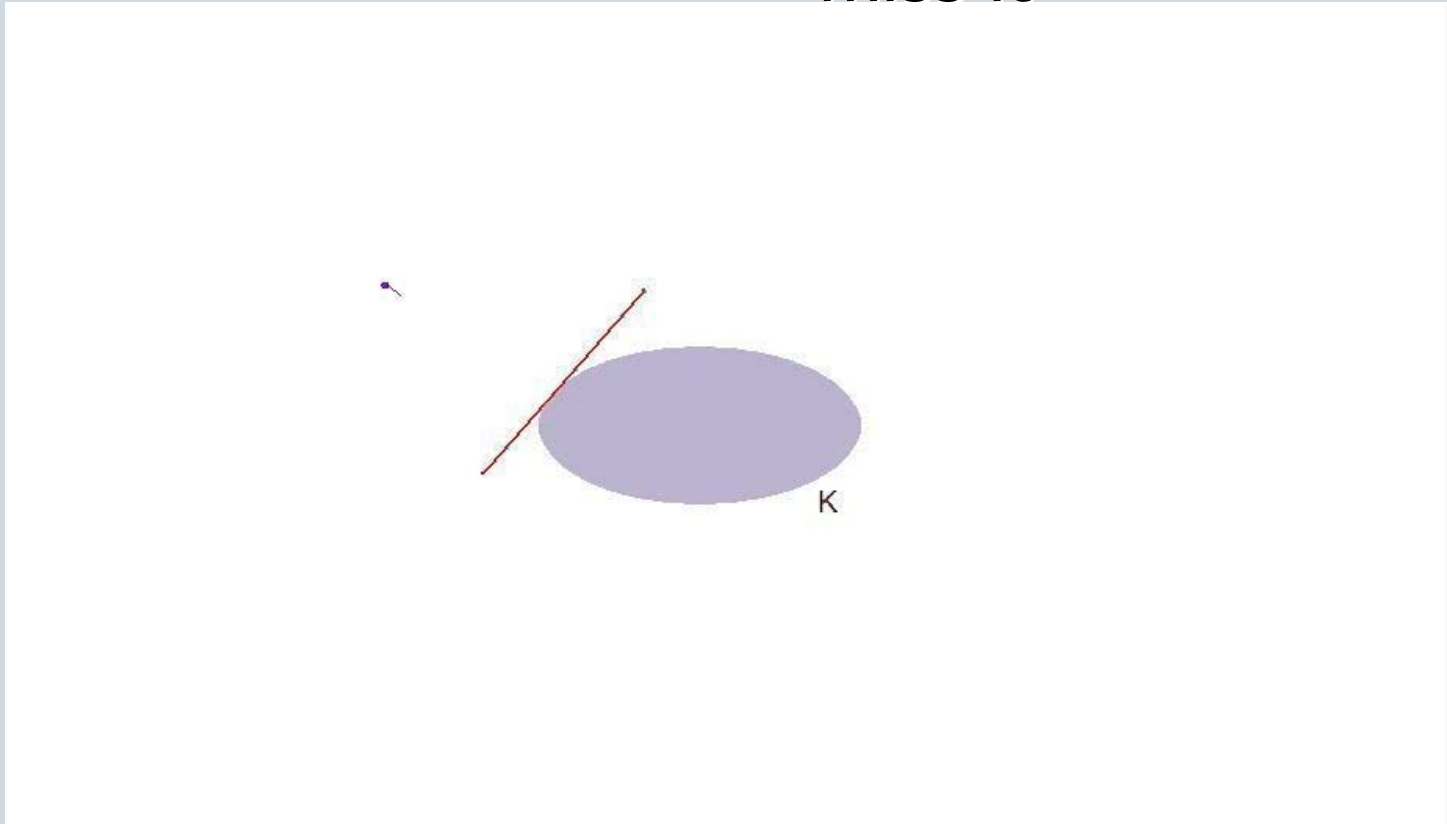
- Most generic context = a Hilbert space X .
[Cojocaru, 2002]

We need:

- K a nonempty, closed, convex subset of X ;
- $F : K \rightarrow X$ a Lipschitz continuous mapping;
- the tangent cone $T_K(x)$ to $x \in K$;
- the projection operator $P : X \rightarrow T_K(x)$;

Super short animation #1

- Do not blink, you may miss it



Projected dynamical systems - II

- A projected differential equation is a discontinuous ODE of the form

$$dx(t)/dt = P_{TK(x(t))}(-F(x(t))) \quad (*)$$

the righthand side is the projection of the velocity field $-F(x(t))$ on the tangent cone to K at $x(t)$

- **Theorem [Cojocaru, '02]:** A PrDE (*) with $x(0) \in K$ and F Lipschitz, has a unique solution in $AC([0, \infty), K)$ through each $x(0) \in K$.
- A PDS is the dynamical system given by the set of trajectories of a PrDE.

Equilibrium analysis for PDS

- An equilibrium for PDS is a point $x \in K$ such that $P_{TK(x)}(-F(x)) = 0$.

VI ★ PDS:

Theorem [Nagurney, '93 – Cojocaru & Isac '02]:

The equilibria of an infinite-dim PDS are solutions to the VI(F,K):

find $x \in K$ such that $\langle F(x), y - x \rangle \geq 0, \forall y \in K$
and vice versa.

$[0, \infty)$ = adjustment scale

Question #2

- Can we model the occurrence of equilibrium states for the transportation problem over a time scale $[0, T]$, assuming the constraints of the problem are time-dependent?

Yes, with a class of time-dependent VI (also called EVI)

Time-dependent VI (EVI)

- EVI: back to '67-'68 works of Stampacchia, Lyons, Brezis;

- Let K in $L^2([0, T], \mathbb{R}^q)$ and the EVI

find $u \in K$ s. t. $\langle \langle F(u), v - u \rangle \rangle \geq 0, \forall v \in K,$
 where

$$K = \{ u \in L^2([0, T], \mathbb{R}^q) \mid \begin{aligned} & \dot{u}(t) \in \mu(t) \text{ a.e. } [0, T]; \\ & \sum_{j=1}^l \alpha_{ji} u_i(t) = \beta_j(t) \text{ a.e. } [0, T], \alpha_{ji} \in \{0, 1\}, \\ & i \in \{1, \dots, q\}, j \in \{1, \dots, l\} \end{aligned} \}$$

$$F : K \rightarrow L^2([0, T], \mathbb{R}^q) \text{ and } \langle \langle \cdot, \cdot \rangle \rangle := \int_0^T \cdot^T \cdot$$

Solutions to EVI

- Theorem [Daniele, Maugeri & al. '98-'03]: If F is pseudo-monotone and hemicontinuous along line segments, then the EVI admits a solution over the constraint set K .
- Theorem [Cojocaru-Daniele-Nagurney '04]: If F is in addition strictly pseudo-monotone, then the solution to the EVI is unique.

Solutions to EVI –cont'd

- Interpretation:
 - at each fixed t in $[0, T]$, the solution(s) of the EVI represent one or more equilibrium states of the problem with the time-dependent constraint set K_t ;
 - as t varies over $[0, T]$, these equilibria describe one (or more) curve(s);
 $[0, T] =$ prediction scale

Question #3

What is then the theoretical interaction of the two infinite-dimensional theories (PDS+EVI) and their associated time scales, and what does it mean for applications?

Double-layered dynamics [Cojocaru-Daniele-Nagurney, '04]

Double-layered dynamics (DLD)

Reminder:

$$K = \{ u \in L^2([0, T], \mathbb{R}^q) \mid \begin{aligned} & \dot{u}(t) \leq \mu(t) \text{ a.e. } [0, T]; \\ & \sum_{j=1}^l \alpha_{ji} u_i(t) = \beta_j(t) \text{ a.e. } [0, T], \alpha_{ji} \in \{0, 1\}, \\ & i \in \{1, \dots, q\}, j \in \{1, \dots, l\} \end{aligned}$$

- Let an EVI as before and consider the PDS on K of $L^2([0, T], \mathbb{R}^q)$, given by:

$$\frac{du(\cdot, \diamond)}{d\diamond} = P_{TK(u(\cdot, \diamond))}(-F(u(\cdot, \diamond))), \quad u(\cdot, 0) \in K$$

(**)

Time scales interpretation-I

Theorem [C-D-N,'04]: The solutions to EVI are the same as the critical points of PDS (***) and vice versa.

For each fixed t in $[0, T]$, \diamond = evolution time of the problem towards one of the equilibria $u^*(t)$, given an initial state in the underlying PDS_t .

Consequence: stability analysis of the curve of equilibria done via infinite-dimensional PDS.

Time scales interpretation-II

- **Theorem [C-D-N]**: If F is strongly pseudo-monotone with degree $\mathfrak{D} < 2$ and Lipschitz on K_t for a.a. fixed t in $[0, T]$, then:
 - ☞ $l_t > 0$ finite, s.t. the unique equilibrium $u^*(t)$ of the PDS_t is reached by the solution $u(t, l_t)$ of the PDS_t , starting at the initial point $u_t^0 \in K_t$.
- $l_t =$ the **adjustment time** to equilibrium $u^*(t)$.

Continuous and Discontinuous curves of equilibria

- **Theorem [C-D-N,'05]:** Whenever the equilibrium constraints depend continuously on t , then the solutions of the EVI are continuous.

- **Additional assumptions:** We consider

$$K = \{ u \in L^2([0, T], \mathbb{R}^q) \mid \text{const}_1 \bullet u(t) \bullet \text{const}_2 \text{ a.e. } [0, T]; \sum u_i(t) = \square_1(t) \text{ a.e. } [0, T], \square_{1i} = 1, i \in \{1, \dots, q\}\}, \text{ where}$$

$$\square_1(t) = \text{is a step function on } [0, T] = [0, t_1] \cup_p (t_{p-1}, t_p]$$

Discontinuous curves of equilibria

Theorem [C-'05]: If the EVI has a unique solution u^* on such K , then it is also a step function, i.e. $u^*(t) = u^*_{p'}$, if t is in $(t_{p-1}, t_p]$, etc.

This follows from the fact that all K_t , with t in $[t_{p-1}, t_p]$ are the same, and so are all the PDS_t

Persistence and adjustment times

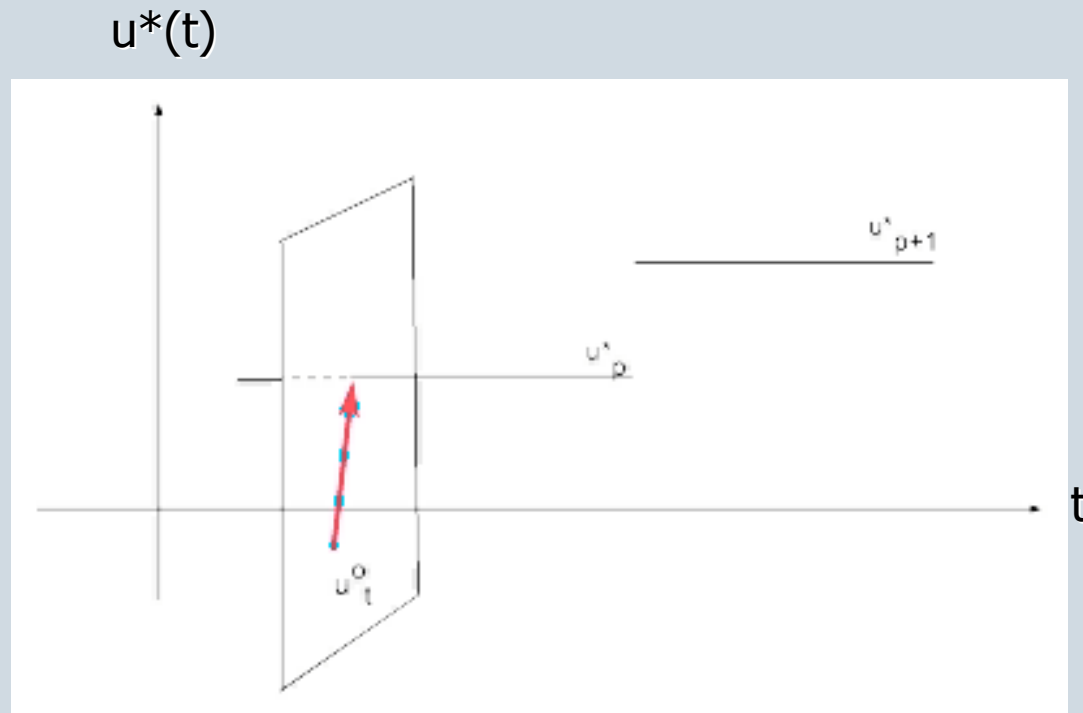
- Fix t in $[0, T]$ w.l.o.g. so that $t \in (t_{p-1}, t_p]$;
- Then $t_p - t$ is called **the persistence time** of the equilibrium u_p^* .

Time scales interpretation: if we start with a disequilibrium state u_t^0 for a PDS_t , then:

- $u^*(t) = u_p^*$ is reached if $t \leq t_p - t$
- $u^*(t) = u_p^*$ is not reached otherwise.

Super-short animation #2

- The process can be visualized here:



DLD model of transportation

- Let $T = 90$ min and $\square_1(t) = 90, 80, \text{ resp. } 60$ cars/min over intervals of 30 min.
- $K = \{ u \in L^2([0, T], \mathbb{R}^2) \mid (1, 0) \bullet u(t) \bullet (140, 140) \text{ a.e. } [0, T]; \sum u_i(t) = \square_1(t) \text{ a.e. } [0, T] \}$
- $F(u_1(t), u_2(t)) = (2u_1(t) - u_2(t) - 2, u_1(t) + u_2(t) - 2)$ are the cost vector functions;
- F is strongly pseudo-monotone with degree $\varphi = 1$ and $\alpha = (1/2)^{1/2}$.

DLD transportation-cont'd

- In this scenario the predicted equilibrium curve is $u^*(t) = (41,49), (41,39),$ respectively $(41,19)$.
- Suppose we start studying the traffic at $t=10\text{min}$ with $u_{10}^0 = (43,47)$.
- Since $u^*(10\text{min})=(41,49)$ will persist for another 20min, then since $l_{10} = 4\text{min} \bullet 20\text{min}$, the problem will reach $u^*(10)$.
- If in turn at $t=10\text{min}$, the traffic is $(73,27)$, then $u^*(10\text{min})$ will not be reached since now $l_{10} > 20$.

References

- Online source:

<http://www.uoguelph.ca/~mcojocar/>

- DLD articles are in JOTA, EJOR, AML, JEDC, Kluwer NOIA series (P. Pardalos & al Edts.)