A Discretization Procedure for Evolutionary Variational Inequalities and Applications to a Financial Model

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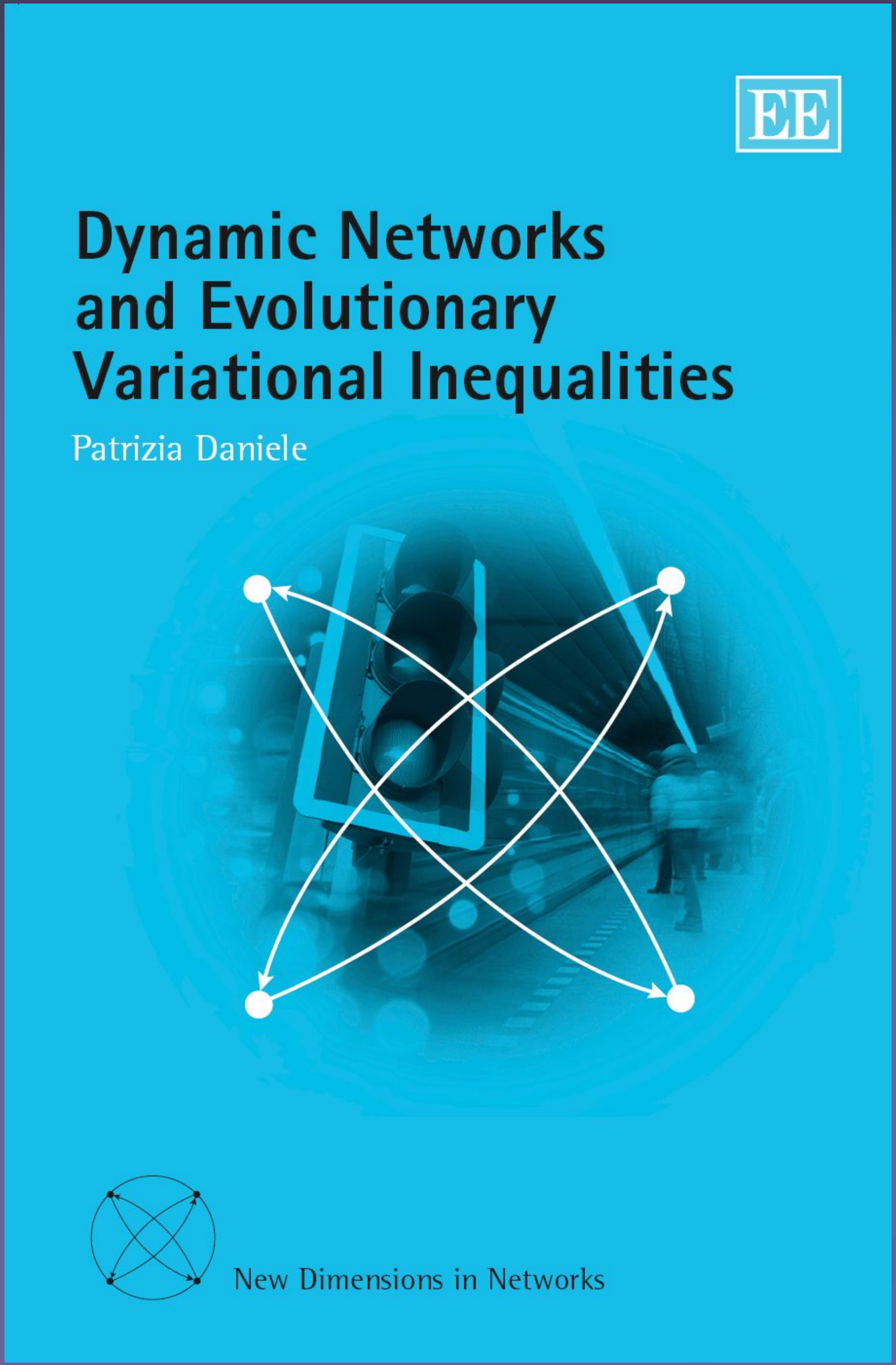
- Presentation of PDS
- Theoretical Results
- Discretization Procedure
- Traffic Network Example
- Evolutionary Financial Model



Edward Elgar Publishing August 2006.

Acknowledgements

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Evolutionary Variational Inequality (EVI)

$$\text{Find } H \in \mathbf{K} : \langle\langle C(H), F - H \rangle\rangle \geq 0 \quad \forall F \in \mathbf{K}$$

$$\Updownarrow$$

$$\int_{[0,T]} \langle C(H(t)), F(t) - H(t) \rangle dt \geq 0$$

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Projected Dynamical System (PDS)

$\phi: \mathbf{R}_+ \times \mathbf{K} \to \mathbf{K}$ solving the initial value problem:

$$\begin{cases} \dot{\phi}(\tau, x) = \Pi_{\mathbf{K}}\left(\phi(\tau, x), -F\left(\phi(\tau, x)\right)\right) \\ \phi(0, x) = x_0 \in \mathbf{K} \end{cases}$$

$\mathbf{K} \subseteq \mathbf{R}^n$: convex polyhedral set;

$F : \mathbf{K} \to \mathbf{R}^n$: Lipschitz continuous function with linear growth;

$\Pi_{\mathbf{K}} : \mathbf{R} \times \mathbf{K} \to \mathbf{R}^n$: Gateaux directional derivative

$$\Pi_{\mathbf{K}}\left(x, -F(x)\right) = \lim_{\delta \to 0^+} \frac{P_{\mathbf{K}}\left(x - \delta F(x)\right) - x}{\delta};$$

$P_{\mathbf{K}} : \mathbf{R}^n \to \mathbf{K}$:projection operator $\left\|P_{\mathbf{K}}(z) - z\right\| = \inf_{y \in \mathbf{K}} \left\|y - z\right\|$;

General Formulation of the set K for traffic network problems, spatial price equilibrium problems, financial equilibrium problems

$$\mathbf{K} = \Big\{ u(t) \in L^2\left([0,T], \mathbf{R}^q\right) : \lambda(t) \le u(t) \le \mu(t) \text{ a.e.in } [0,T];$$

$$\left. \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i = 1,\dots,q, j = 1,\dots,l \right\}$$

$$\lambda, \mu, \rho \in L^p\left([0,T], \mathbf{R}^q\right) : \text{given functions}$$

Preliminary Definitions

- $M \subset H \Rightarrow M^0 = \left\{ \xi \in H : \langle \xi, x \rangle \leq 1, \ \forall x \in M \right\}$

- C cone $\Rightarrow C^0 = C^- = \left\{ \xi \in H : \langle \xi, x \rangle \leq 0, \ \forall x \in C \right\}$

- $\mathbf{K} \subset H$ nonempty, closed, convex $\Rightarrow$

  $T_{\mathbf{K}}(x) = \overline{\bigcup_{\lambda > 0} \lambda \left(\mathbf{K} - x \right)} =$ support cone

- $N_{\mathbf{K}}(x) = \left\{ \xi \in H : \langle \xi, z - x \rangle \leq 0, \ \forall z \in \mathbf{K} \right\} =$

  normal cone to $\mathbf{K}$ at x

- **Prop. 1:** $\left(T_{\mathbf{K}}(x)\right)^{0} = N_{\mathbf{K}}(x) = \left(T_{\mathbf{K}}(x)\right)^{-}$

- **Theor. 1:** $P_{\mathbf{K}}\left(x+\lambda h\right) = x + \lambda P_{T_{\mathbf{K}}(x)} h + o(\lambda)$

  $\forall x, h, \lambda > 0$

- **Cor. 1:** If $\Pi_{\mathbf{K}}\left(x,h\right) = \lim_{\lambda \to 0^{+}} \dfrac{P_{\mathbf{K}}\left(x+\lambda h\right) - x}{\lambda}$,

  then $\Pi_{\mathbf{K}}\left(x,h\right) = P_{T_{\mathbf{K}}(x)} h.$

- $n_{\mathbf{K}}(x)=\left\{v:\|v\|=1 \text{ and } \langle v, x-y\rangle \leq 0,\ \forall y\in\mathbf{K}\right\}=$

  set of unit inward normals to $\mathbf{K}$ at x

- **Prop. 2:** $n_{\mathbf{K}}(x)=\partial B(0,1)\cap-\left(T_{\mathbf{K}}(x)\right)^{0}$

- $\operatorname{qi}\mathbf{K}=\left\{x\in\mathbf{K}:T_{\mathbf{K}}(x)=H\right\}$

- $\operatorname{qbdry}\mathbf{K}=\mathbf{K}\setminus\operatorname{qi}\mathbf{K}$

- **Prop. 3:** $x\in\operatorname{qbdry}\mathbf{K}\Leftrightarrow n_{\mathbf{K}}(x)\neq\varnothing$

- **Theor. 2:**

$$x \in \operatorname{qi}\mathbf{K} \Rightarrow \Pi_{\mathbf{K}}(x,h) = h,\ \forall h \in H;$$

$$x \in \operatorname{qbdry}\mathbf{K} \Rightarrow \forall v \in H \setminus T_{\mathbf{K}}(x), \exists n^*(x) \in n_{\mathbf{K}}(x):$$

$$\beta(x) = -\left\langle v, n^*(x) \right\rangle > 0,$$

$$\Pi_{\mathbf{K}}\left(x,v\right) = v + \beta(x)\, n^*(x).$$

- **Cor. 2:**

$$\Pi_{\mathbf{K}}\left(x,v\right) = P_{v-N_{\mathbf{K}}(x)}\left(0\right) = \left(v - N_{\mathbf{K}}(x)\right)^{\#} \quad \forall v \in H.$$

Cor. 2 $\Rightarrow \exists\, !\, n_x : \Pi_{\mathbf{K}}\left(x, -F\left(x\right)\right) = -F\left(x\right) - n_x$

where $n_x = 0$ if $x \in \operatorname{qi}\mathbf{K}$

- **Remark 1:** Cor. 2 implies

$$\frac{d\,x(t)}{d\,t} = \Pi_{\mathbf{K}}\left(x, -F(x)\right) = P_{-F(x)-N_{\mathbf{K}}(x)}(0) =$$

$$\left\{ \bar{v} \in -\left(F(x) + N_{\mathbf{K}}(x)\right) : \left\| \bar{v} \right\| = \min_{y \in -\left(F(x)+N_{\mathbf{K}}(x)\right)} \| y \| \right\}$$

$$\begin{cases} \dfrac{dx(t)}{d\,t} = \Pi_{\mathbf{K}}\left(x(t), -F\left(x(t)\right)\right) \\ x(0) = x_0 \in \mathbf{K} \end{cases} \Leftrightarrow \begin{array}{l} \text{slow solution to} \\ \dot{x}(t) \in -\left(N_{\mathbf{K}}\left(x(t)\right) + F\left(x(t)\right)\right) \\ x(0) = x_0 \end{array}$$

Remark 2:

$$\text{Find } u \in \mathbf{K}: \left\langle\left\langle F(u), v-u \right\rangle\right\rangle \geq 0, \ \forall v \in \mathbf{K}$$

$$\text{where} \left\langle\left\langle \Phi, u \right\rangle\right\rangle = \int_0^T \left\langle \Phi(t), u(t) \right\rangle dt,$$

$$\Phi \in L^2\left(\left[0,T\right], \mathbf{R}^q\right)^*, \ \ u \in L^2\left(\left[0,T\right], \mathbf{R}^q\right)$$

$$\Updownarrow$$

$$\text{Find } u \in \mathbf{K}: \left\langle F\left(u(t)\right), v(t) - u(t) \right\rangle \geq 0,$$

$$\forall v \in \mathbf{K}, \text{a.e. in } \left[0,T\right]$$

Discretization Procedure

Partitions of $[0,T]$:

$$\pi_n = \left(t_n^0, t_n^1, \ldots, t_n^{N_n}\right), 0 = t_n^0 < t_n^1 < \ldots < t_n^{N_n} = T$$

$$\delta_n = \max\left\{t_n^j - t_n^{j-1} : j = 1, \ldots, N_n\right\}, \lim_n \delta_n = 0.$$

$$P_n\left([0,T], \mathbf{R}^m\right) = \left\{v \in L^\infty\left([0,T], \mathbf{R}^m\right) : v_{\left(t_n^{j-1}, t_n^j\right]} = v_j \in \mathbf{R}^m, \right.$$
$$\left. j = 1, \ldots, N_n\right\}$$

$$\mu_n v_{\left(t_n^{j-1}, t_n^j\right]} := \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} v(s)\, ds$$

$$\mathbf{K} = \Big\{ F(t) \in L^2\left([0,T], \mathbf{R}^m\right) : \lambda \le F(t) \le \nu \text{, a.e. in} [0,T],$$

$$\Phi F(t) = \rho,\ \ \lambda, \nu \ge 0 \Big\}$$

$$C\big[t, F(t)\big] = A(t) F(t) + B(t),$$

$$A(t) \in L^\infty\left([0,T], \mathbf{R}^{m^2}\right),\ B(t) \in L^2\left([0,T], \mathbf{R}^m\right)$$

$$\int_0^T \left\langle A(t)H(t)+B(t),\, F(t)-H(t)\right\rangle dt$$

$$=\sum_{j=1}^{N_n}\int_{t_n^{j-1}}^{t_n^j} \left\langle A(t)H(t)+B(t),\, F(t)-H(t)\right\rangle dt$$

$$H_j^n(t)\in\mathbf{K}:$$

$$\int_{t_n^{j-1}}^{t_n^j} \left\langle A(t)H_j^n(t)+B(t),\, F_j^n(t)-H_j^n(t)\right\rangle dt\geq 0,$$

$$\forall F_j^n(t)\in\mathbf{K}$$

Finite-dimensional Variational Inequality:

$$H_j^n \in \mathbf{K}_m : \left\langle A_j^n H_j^n + B_j^n , F_j^n - H_j^n \right\rangle \geq 0, \ \forall F_j^n \in \mathbf{K}_m$$

where

$$A_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} A(t)\,dt; \quad B_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} B(t)\,dt.$$

$$H_n(t) = \sum_{j=1}^{N_n} \chi\left(t_n^{j-1}, t_n^j\right) H_j^n$$ piecewise constant

approximation to the solution of EVI

Theor. 1

$A(t)$ positive definite a.e. in $[0,T] \Rightarrow$ the set $U = \{H_n\}_{n\in\mathbb{N}}$ (weakly) compact, with feasible cluster points solutions to the EVI.

$$\mathbf{K} = \Big\{ F(t) \in L^2\left([0,T], \mathbf{R}^m\right) : \lambda(t) \le F(t) \le \nu(t) \text{ a.e.in } [0,T];$$
$$\lambda(t), \nu(t) \ge 0,\ \Phi F(t) = \rho(t) \text{ a.e. in } [0,T] \Big\}$$

$$\pi_n : \mathbf{K}_j^n = \Big\{ F_j \in \mathbf{R}^m : \overline{\lambda}_{j,n} \le F_j \le \overline{\nu}_{j,n} \text{ a.e.in } \left(t_{j-1}, t_j\right);$$
$$\Phi F_j = \overline{\rho}_{j,n} \text{ a.e. in } \left(t_{j-1}, t_j\right) \Big\}$$

Lemma 1 $\mathbf{K}_n = \bigcap_j \mathbf{K}_j^n \to \mathbf{K}$

Initial problem:

$$H(t) \in \mathbf{K}(t)$$

$$\int_0^T \left\langle C\left[t, H(t)\right], F(t) - H(t) \right\rangle dt \geq 0 \quad \forall F(t) \in \mathbf{K}(t)$$

Theor. 2

$A(t)$ positive definite a.e. in $\left[0,T\right] \Rightarrow H_n(t) = \sum_{j=1}^{N_n} \chi\left(t_n^{j-1}, t_n^j\right) H_j^n$

has (weakly) cluster points solutions to the initial problem.

Traffic Network Example

$$A(t)=\begin{bmatrix} 2+t & 0 \\ 0 & 1 \end{bmatrix},\ F(t)=\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix},\ B(t)=\begin{bmatrix} -\frac{3}{2} \\ 2 \\ -1 \end{bmatrix},$$

$$\mathbf{K}=\{F\in L^2([0,2],\mathbf{R}^2):0\le F_1(t)\le t;\ 0\le F_2(t)\le 3;$$

$$F_1(t)+F_2(t)=t \text{ a.e.in } [0,2]\}$$

$$0<\frac{2}{n}<\frac{4}{n}<\ldots<(j-1)\frac{2}{n}<j\frac{2}{n}<\ldots<(n-1)\frac{2}{n}<2$$

$$\left\langle A_j^n H_j^n + B_j^n, F_j^n - H_j^n \right\rangle \geq 0 \quad \forall F_j^n \in \mathbf{K}_j^n$$

$$\mathbf{K}_j^n = \left\{ F_j^n \in \mathbf{R}^2 : 0 \leq F_{j1}^n \leq \frac{2j-1}{n},\ 0 \leq F_{j2}^n \leq 3,\ F_{j1}^n + F_{j2}^n = \frac{2j-1}{n} \right\}$$

$$\left[\left(2 + \frac{2j-1}{n} \right) H_{j1}^n - \frac{3}{2} \right] \left(F_{j1}^n - H_{j1}^n \right) + \left(H_{j2}^n - 1 \right) \left(F_{j2}^n - H_{j2}^n \right) \geq 0 \quad \forall F_j^n \in \mathbf{K}_j^n$$

$$F_{j2}^n = \frac{2j-1}{n} - F_{j1}^n$$

$$\left[\left(2 + \frac{2j-1}{n} \right) H_{j1}^n - \frac{3}{2} - \frac{2j-1}{n} + H_{j1}^n + 1 \right] \left(F_{j1}^n - H_{j1}^n \right) \geq 0$$

$$H_{j1}^{n}=\begin{cases}\dfrac{2j-1}{n} & \text{if}\quad 1\le j<\dfrac{1}{2}\left(1+\dfrac{n}{2+\sqrt{6}}\right)\\[2ex] \dfrac{4j-2+n}{2\left(3n+2j-1\right)} & \text{if}\quad j>\dfrac{1}{2}\left(1+\dfrac{n}{2+\sqrt{6}}\right)\end{cases}$$

$$H_{j2}^{n}=\begin{cases}0 & \text{if}\quad 1\le j<\dfrac{1}{2}\left(1+\dfrac{n}{2+\sqrt{6}}\right)\\[2ex] \dfrac{-n^{2}+4\left(2j-1\right)n+2\left(2j-1\right)^{2}}{2n\left(3n+2j-1\right)} & \text{if}\quad j>\dfrac{1}{2}\left(1+\dfrac{n}{2+\sqrt{6}}\right)\end{cases}$$

$$H_{n}\left(t\right)=\sum_{j=1}^{n}\chi\left[\left(j-1\right)\frac{2}{n},\, j\frac{2}{n}\right]H_{j}^{n}$$

Direct Method

$$\left[(2+t)H_1(t)-\frac{3}{2}\right]\left[F_1(t)-H_1(t)\right]+\left[H_2(t)-1\right]\left[F_2(t)-H_2(t)\right]\geq 0$$

$$\forall F(t)\in\mathbf{K}=\left\{F(t)\in L^2\left([0,2],\mathbf{R}^2\right):\ 0\leq F_1(t)\leq t;\right.$$

$$\left.0\leq F_2(t)\leq 3;\ F_1+F_2(t)=t,\ \text{a.e. in } [0,2]\right\}$$

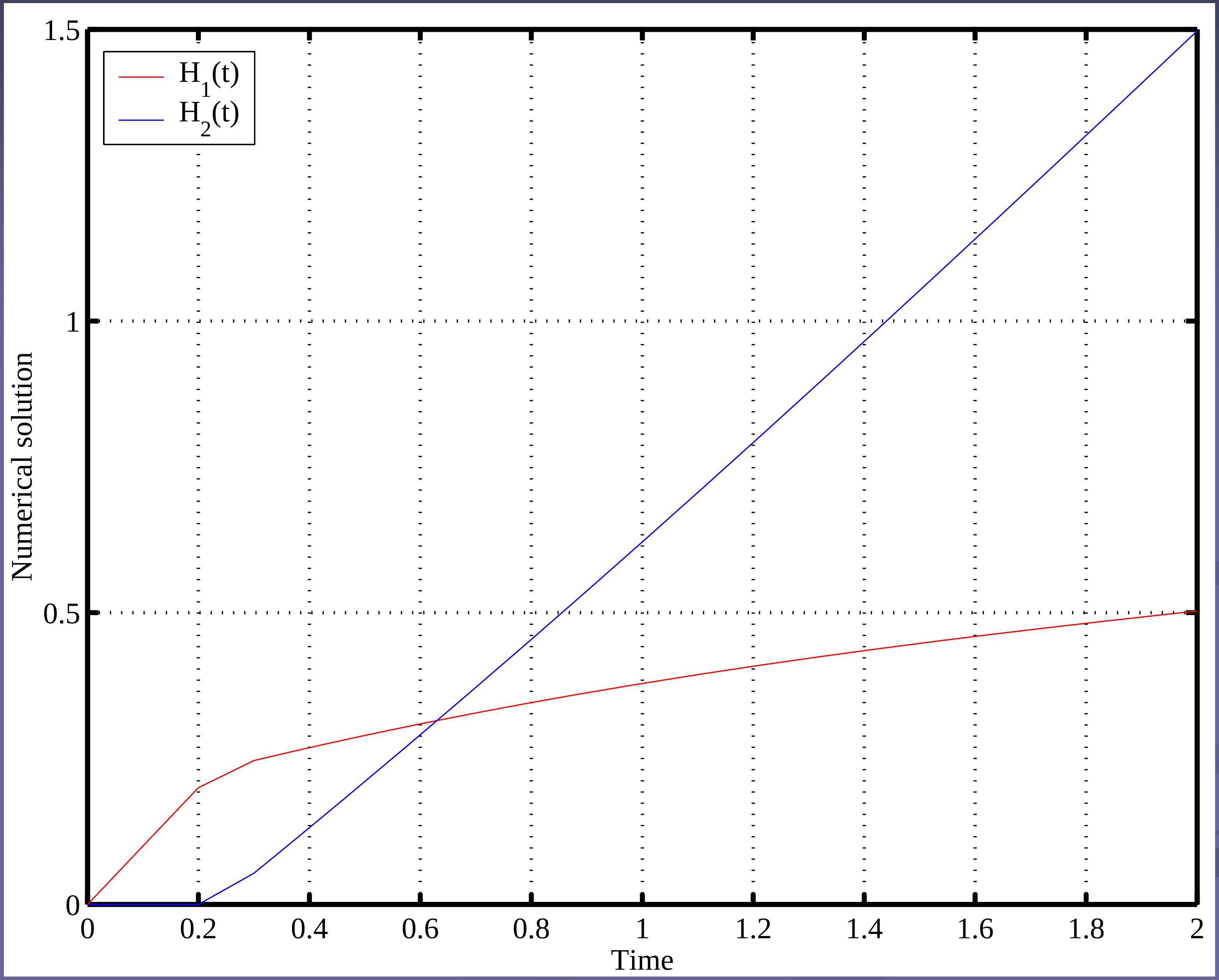
$$F_2(t)=t-F_1(t)\ \Rightarrow\ \widetilde{\mathbf{K}}=\left\{F_1(t)\in L^2\left([0,2],\mathbf{R}\right):0\leq F_1(t)\leq t\right\}$$

$$\left[3(2+t)H_1(t)-t-\frac{1}{2}\right]\left[F_1(t)-H_1(t)\right]\geq 0\ \ \forall F_1(t)\in\widetilde{\mathbf{K}}$$

$$H_1(t)=\begin{cases} t & \text{if } 0\le t\le \frac{\sqrt{6}}{2}-1 \\ \frac{2t+1}{2t+6} & \text{if } \frac{\sqrt{6}}{2}-1<t\le 2 \end{cases}$$

$$H_2(t)=\begin{cases} 0 & \text{if } 0\le t\le \frac{\sqrt{6}}{2}-1 \\ \frac{2t^2+4t-1}{2t+6} & \text{if } \frac{\sqrt{6}}{2}-1<t\le 2 \end{cases}$$

$$\left(H_1^n(t),H_2^n(t)\right)\to\left(H_1(t),H_2(t)\right)$$



$H_1(t)$
$H_2(t)$
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time
Numerical solution

Financial Markets

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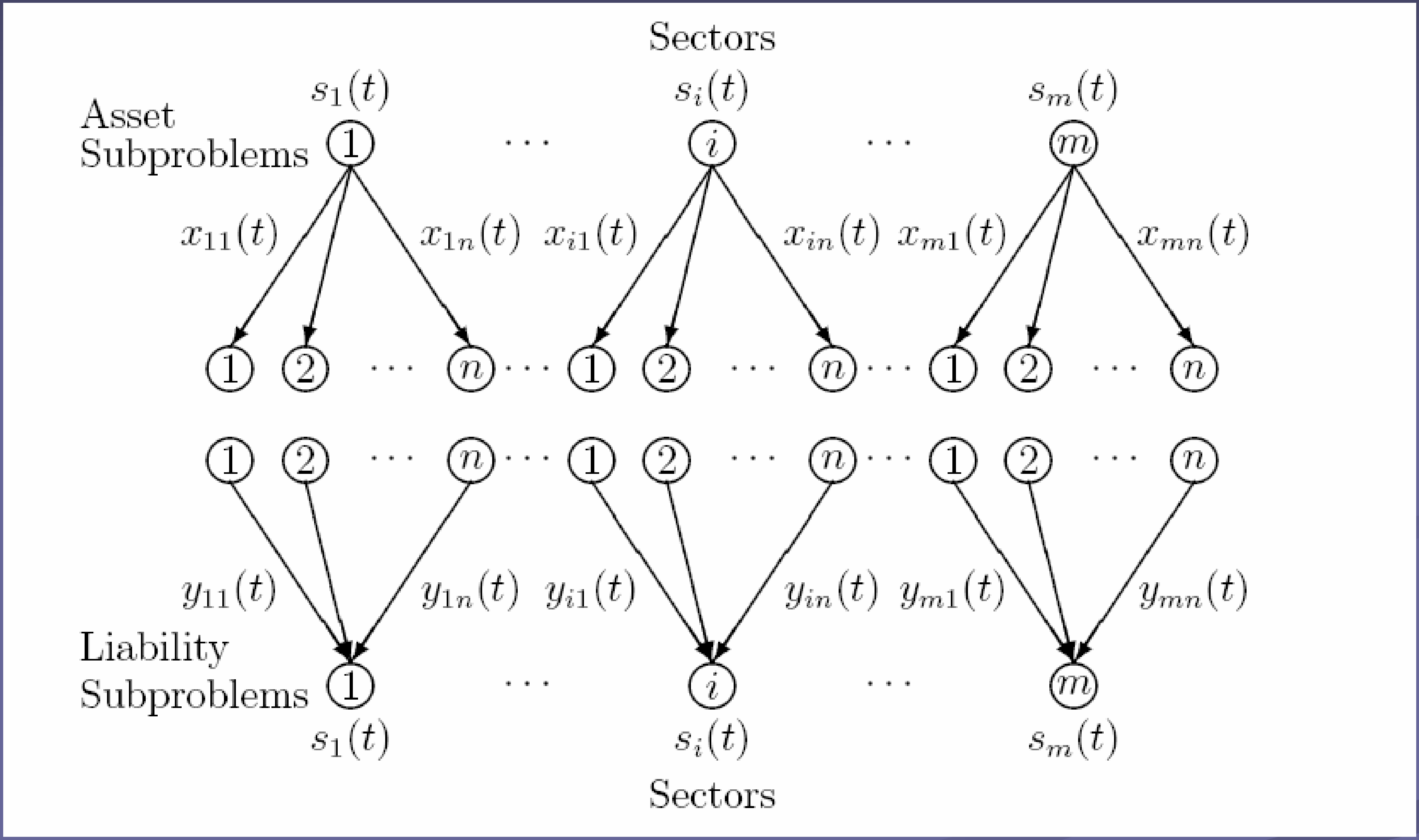
- m **sectors**: households, domestic business, banks, financial institutions, state and local governments, …
- n **financial instruments**: mortgages, mutual funds, savings deposits, money market funds, …
- $s_i(t)$: **total financial volume** held by sector i at time t
- $x_{ij}(t)$: amount of instrument j held as an **asset** in sector i's portfolio
- $y_{ij}(t)$: amount of instrument j held as a **liability** in sector i's portfolio



Sectors
$s_1(t)$
$s_i(t)$
$s_m(t)$
Asset Subproblems
1
i
m
$x_{11}(t)$
$x_{1n}(t)$
$x_{i1}(t)$
$x_{in}(t)$
$x_{m1}(t)$
$x_{mn}(t)$
1 2 n 1 2 n 1 2 n
1 2 n 1 2 n 1 2 n
$y_{11}(t)$
$y_{1n}(t)$
$y_{i1}(t)$
$y_{in}(t)$
$y_{m1}(t)$
$y_{mn}(t)$
Liability Subproblems
1
i
m
$s_1(t)$
$s_i(t)$
$s_m(t)$
Sectors

$$Q^i(t) = \begin{bmatrix} Q^i_{11}(t) & Q^i_{12}(t) \\ Q^i_{21}(t) & Q^i_{22}(t) \end{bmatrix}:$$ **variance-covariance matrix**

$$\begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i(t) \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}:$$ **aversion to the risk** at time *t*

$r_j(t)$: **price of instrument** *j* at time *t*

$$P_i = \left\{ \left(x_i(t), y_i(t) \right) \in L^2 \left([0,T], \mathbf{R}^{2n} \right) : \sum_{j=1}^{n} x_{ij}(t) = s_i(t), \right.$$

$$\left. \sum_{j=1}^{n} y_{ij}(t) = s_i(t),\ x_{ij}(t) \geq 0, y_{ij}(t) \geq 0, \text{a.e.in } [0,T] \right\}:$$

set of feasible assets and liabilities

Theor. 1 (Variational Formulation)

$$\left(x^*(t), y^*(t), r^*(t)\right) \in \prod_{i=1}^{m} P_i \times L^2\left([0,T], \mathbf{R}_+^n\right)$$

evolutionary financial equilibrium

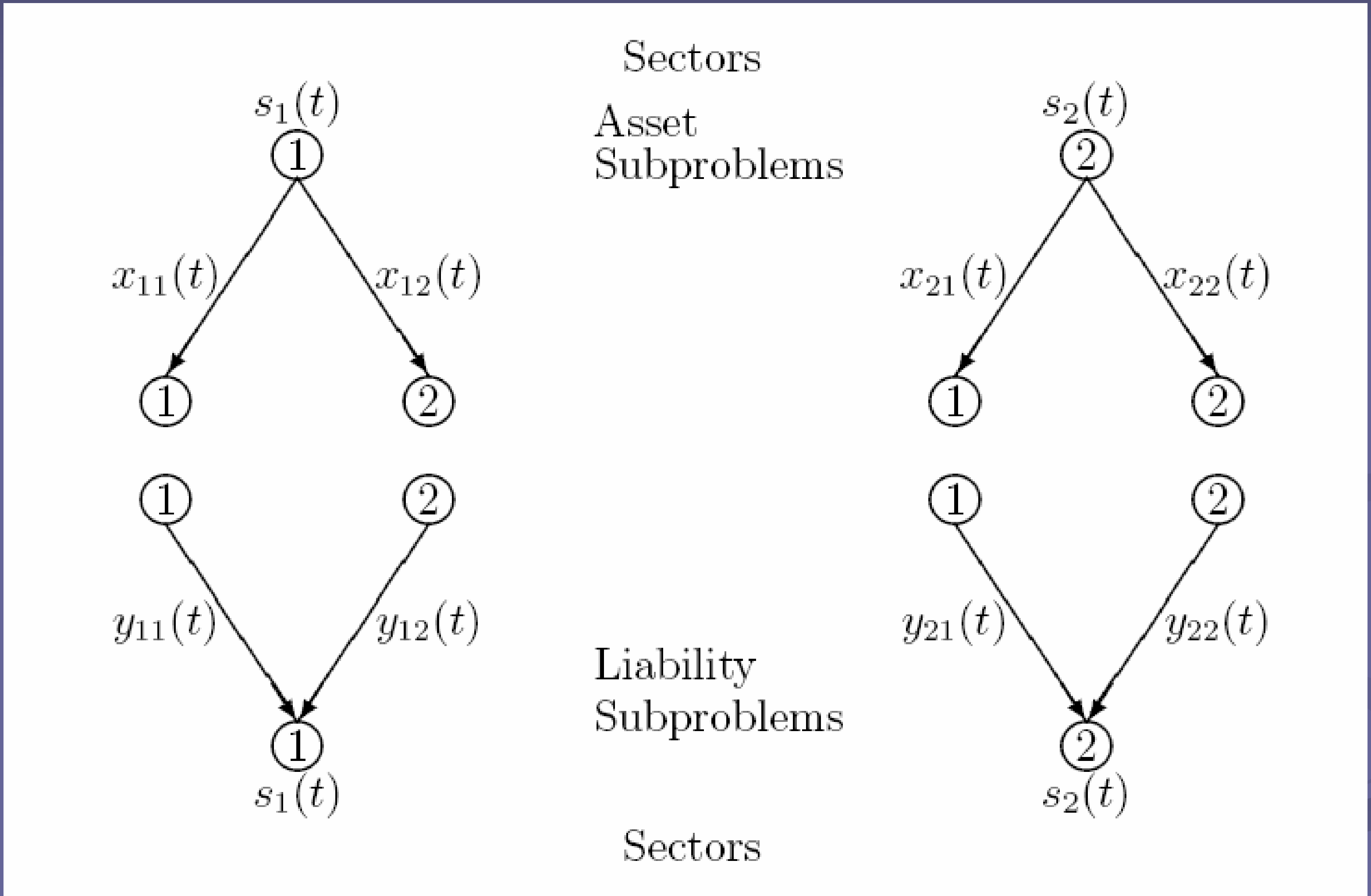
$$\int_0^T \left\{ \sum_{i=1}^{m} \left[2\left[Q_{11}^i(t)\right]^T x_i^*(t) + 2\left[Q_{21}^i(t)\right]^T y_i^*(t) - r^*(t) \right] \times \left[x_i(t) - x_i^*(t)\right] \right.$$

$$+ \sum_{i=1}^{m} \left[2\left[Q_{12}^i(t)\right]^T x_i^*(t) + 2\left[Q_{22}^i(t)\right]^T y_i^*(t) + r^*(t) \right] \times \left[y_i(t) - y_i^*(t)\right]$$

$$\left. + \sum_{i=1}^{m} \left(x_i^*(t) - y_i^*(t)\right) \times \left[r(t) - r^*(t)\right] \right\} dt \geq 0,$$

$$\forall \left(x(t), y(t), r(t)\right) \in \prod_{i=1}^{m} P_i \times L^2\left([0,T], \mathbf{R}_+^n\right)$$

Numerical Example



$$Q^1(t)=\begin{bmatrix} 1 & 0 & -0.5t & 0 \\ 0 & 1 & 0 & 0 \\ -0.5t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ e } Q^2(t)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.5t & 0 \\ 0 & -0.5t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P=\Big\{\big(x_{11}(t),x_{12}(t),x_{21}(t),x_{22}(t),y_{11}(t),y_{12}(t),$$
$$y_{21}(t),y_{22}(t),r_1(t),r_2(t)\big)\in L^2\left([0,1],\mathbf{R}_+^{10}\right):$$
$$x_{11}(t)+x_{12}(t)=t,\ \ y_{11}(t)+y_{12}(t)=t,$$
$$x_{21}(t)+x_{22}(t)=2,\ y_{21}(t)+y_{22}(t)=2,$$
$$4+2t\le r_1(t)\le 8,\ \ 0\le r_2(t)\le t\Big\}$$

$$\left[2x_{11}^{*}(t)-t\,y_{11}^{*}(t)-r_1^{*}(t)\right]\cdot\left[x_{11}(t)-x_{11}^{*}(t)\right]+\left[2x_{12}^{*}(t)-r_2^{*}(t)\right]\cdot\left[x_{12}(t)-x_{12}^{*}(t)\right]+$$

$$+\left[2y_{11}^{*}(t)-t\,x_{11}^{*}(t)+r_1^{*}(t)\right]\cdot\left[y_{11}(t)-y_{11}^{*}(t)\right]+\left[2y_{12}^{*}(t)+r_2^{*}(t)\right]\cdot\left[y_{12}(t)-y_{12}^{*}(t)\right]+$$

$$+\left[2y_{21}^{*}(t)-t\,x_{22}^{*}(t)+r_1^{*}(t)\right]\cdot\left[y_{21}(t)-y_{21}^{*}(t)\right]+\left[2y_{22}^{*}(t)+r_2^{*}(t)\right]\cdot\left[y_{22}(t)-y_{22}^{*}(t)\right]+$$

$$+\left[x_{11}^{*}(t)+x_{21}^{*}(t)-y_{11}^{*}(t)-y_{21}^{*}(t)\right]\cdot\left[r_1(t)-r_1^{*}(t)\right]+$$

$$+\left[x_{12}^{*}(t)+x_{22}^{*}(t)-y_{12}^{*}(t)-y_{22}^{*}(t)\right]\cdot\left[r_2(t)-r_2^{*}(t)\right]\geq 0$$

$$\forall\left(x(t),y(t),r(t)\right)\in P$$

$$x_{12}(t)=t-x_{11}(t),\quad x_{22}(t)=2-x_{21}(t),$$

$$y_{12}(t)=t-y_{11}(t),\quad y_{22}(t)=2-y_{21}(t)$$

$$\widetilde{P}=\Big\{\left(x_{11}(t),\,x_{21}(t),y_{11}(t),y_{21}(t),r_1(t),r_2(t)\right)\in L^2\left([0,1],\mathbf{R}_+^6\right):$$

$$0\leq x_{11}(t)\leq t,\ 0\leq y_{11}(t)\leq t,\ 0\leq x_{21}(t)\leq 2, 0\leq y_{21}(t)\leq 2,$$

$$4+2t\leq r_1(t)\leq 8,\quad 0\leq r_2(t)\leq t\Big\}$$

$$
\begin{aligned}
&\left[4x_{11}^{*}(t)-t\,y_{11}^{*}(t)-2t-r_1^{*}(t)+r_2^{*}(t)\right]\cdot\left[x_{11}(t)-x_{11}^{*}(t)\right]\\
&+\left[4x_{21}^{*}(t)+t\,y_{21}^{*}(t)-4-r_1^{*}(t)+r_2^{*}(t)\right]\cdot\left[x_{21}(t)-x_{21}^{*}(t)\right]\\
&+\left[x_{11}^{*}(t)+x_{21}^{*}(t)-y_{11}^{*}(t)-y_{21}^{*}(t)\right]\cdot\left[r_1(t)-r_1^{*}(t)\right]\\
&-\left[x_{11}^{*}(t)+x_{21}^{*}(t)-y_{11}^{*}(t)-y_{21}^{*}(t)\right]\cdot\left[r_2(t)-r_2^{*}(t)\right]\geq 0
\end{aligned}
$$

$$
\forall\left(\tilde{x}(t),\tilde{y}(t),\tilde{r}(t)\right)\in\tilde{P}
$$

$r_1(t)=r_1^{*}(t)$ e $r_2(t)=r_2^{*}(t)$

$$
\begin{cases}
\Gamma_1 = 0 \Leftrightarrow x_{11}^{*}(t) = \dfrac{2t^2 + 8t - (t-4)\left(r_1^{*}(t) - r_2^{*}(t)\right)}{16 - t^2} \\
\Gamma_2 = 0 \Leftrightarrow y_{11}^{*}(t) = \dfrac{2t^2 + 8t - (t-4)\left(r_1^{*}(t) - r_2^{*}(t)\right)}{16 - t^2} \\
\Gamma_3 = 0 \Leftrightarrow x_{21}^{*}(t) = \dfrac{-2t^2 - 4t + 16 + (t+4)\left(r_1^{*}(t) - r_2^{*}(t)\right)}{16 - t^2} \\
\Gamma_4 = 0 \Leftrightarrow y_{21}^{*}(t) = \dfrac{4t + 16 + (4-t)\left(r_1^{*}(t) - r_2^{*}(t)\right)}{16 - t^2} \\
\left(x_{11}^{*}(t), x_{12}^{*}(t), y_{11}^{*}(t), y_{12}^{*}(t), r_1^{*}(t), r_2^{*}(t)\right) \in \tilde{P}
\end{cases}
\quad \Rightarrow \quad \varnothing
$$

$$
x_{11}^{*}(t) = t,\; x_{21}^{*}(t) = 2,\; y_{11}^{*}(t) = 0,\; y_{21}^{*}(t) = 0
$$

$$(t+2)\left(r_1(t)-r_1^*(t)\right)-(t+2)\left(r_2(t)-r_2^*(t)\right)\geq 0$$

$$\forall\ 4+2t\leq r_1(t)\leq 8,\quad 0\leq r_2(t)\leq t$$

$$r_2(t)=r_2^*(t)\ \Rightarrow (t+2)\left(r_1(t)-r_1^*(t)\right)\geq 0\ \ \forall\ 4+2t\leq r_1(t)\leq 8$$

$$\Downarrow$$

$$r_1^*(t)=4+2t$$

$$r_1(t)=r_1^*(t)\ \Rightarrow -(t+2)\left(r_2(t)-r_2^*(t)\right)\geq 0\ \ \forall\ 0\leq r_2(t)\leq t$$

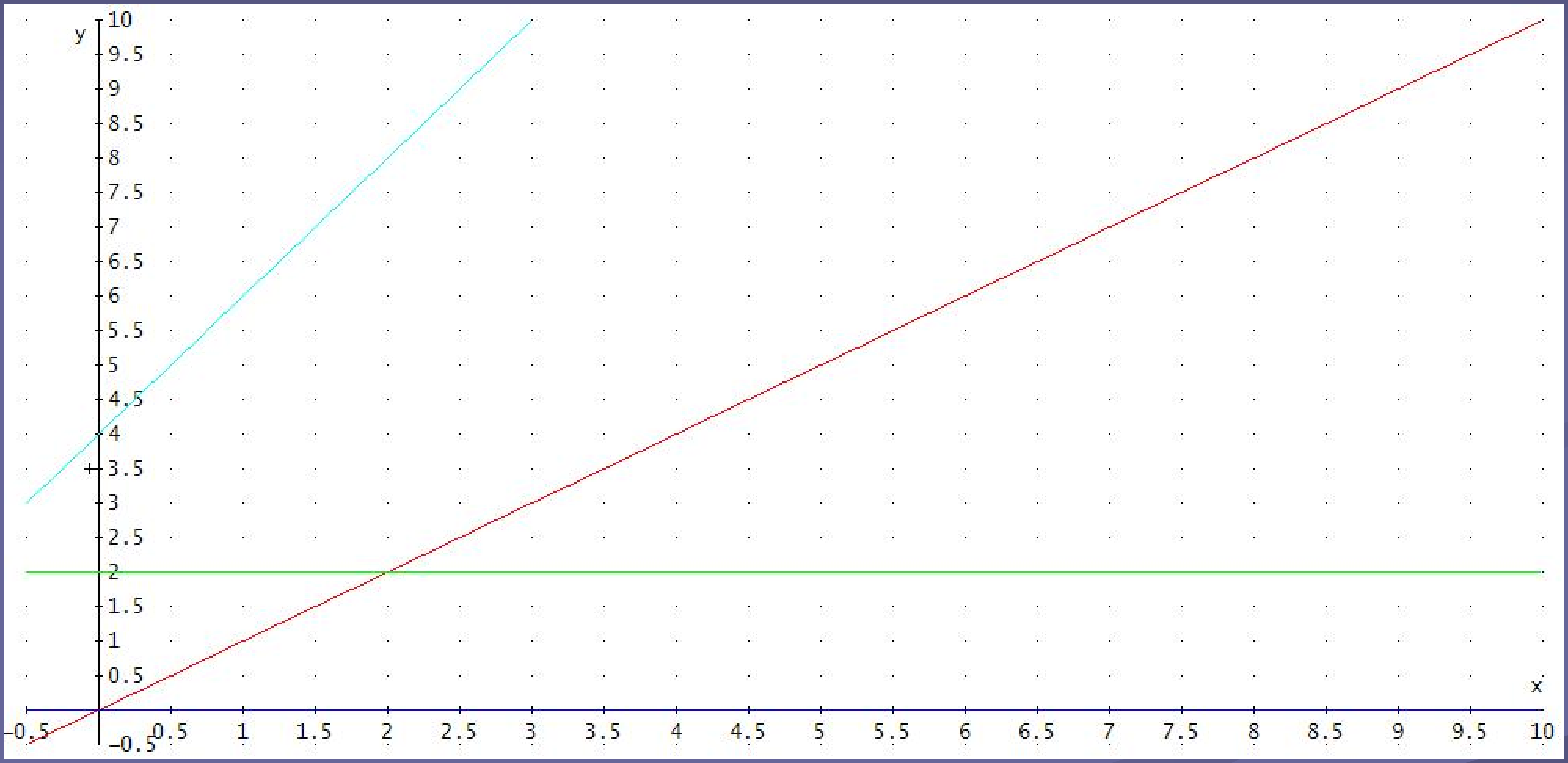
$$\Downarrow$$

$$r_2^*(t)=t$$

Final Solution:

$$\begin{cases} x_{11}^{*}(t)=t, & x_{12}^{*}(t)=0, \\ x_{21}^{*}(t)=2, & x_{22}^{*}(t)=0, \\ y_{11}^{*}(t)=0, & y_{12}^{*}(t)=t, \\ y_{21}^{*}(t)=0, & y_{22}^{*}(t)=2, \\ r_{1}^{*}(t)=4+2t, & r_{2}^{*}(t)=t. \end{cases}$$

Figure



y
x
10
9.5
9
8.5
8
7.5
7
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
−0.5
−0.5
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10