Projected Dynamical Systems and Evolutionary Variational Inequalities with Applications to Dynamic Traffic Networks

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Outline

We present here brief introductions to projected dynamical systems and evolutionary variational inequalities and continue with highlighting the way the two theories intertwine to reveal novel ways of interpreting applied problems. We shall proceed as below:

- Projected differential equations (PrDE) and projected dynamical systems (PDS);
- Evolutionary variational inequalities (EVI);
- How the PDS and EVI mesh;
- Application of the two theories to traffic networks.

PrDE and PDS - I

The most general mathematical context to date in which we can define a projected differential equation (PrDE) and, consequently, a projected dynamical system (PDS), is that of a Hilbert space X of arbitrary (finite or infinite) dimension.

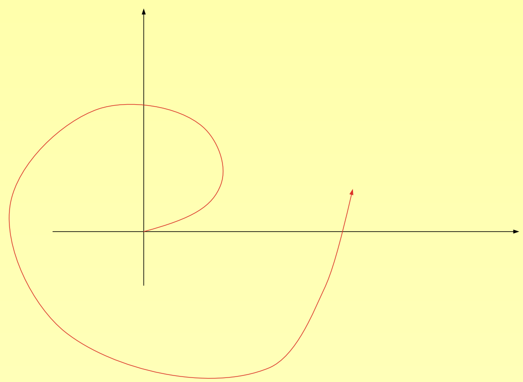
Suppose we have $K \subset X$, a nonempty, closed convex subset in a Hilbert space X. Let $F : K \to X$ a Lipschitz continuous mapping. It is well-known that the ODE:

$$\frac{dx(\tau)}{d\tau} = -F(x(\tau)),\ x(0) \in K$$

has solutions in a suitable class of functions; here that class will be that of absolutely continuous functions $AC([0, \infty), X)$.

Let us define a PrDE on an example, "with drawings":

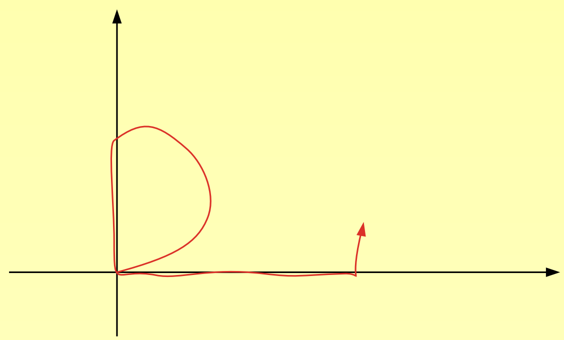
Suppose $X := \mathbb{R}^2$, $K := \mathbb{R}^2_+$ and suppose that the image to the left represents a trajectory of the equation $\dfrac{dx(\tau)}{d\tau} = -F(x(\tau))$, starting in $\mathbb{R}^2$.



A PrDE describes the control problem:

$$\frac{d}{d\tau}(x(\tau)) = -F(x(\tau)), x(0) \in \mathbb{R}^2 \text{ such that } x(\tau) \in \mathbb{R}^2_+,$$

as shown in the figure below:

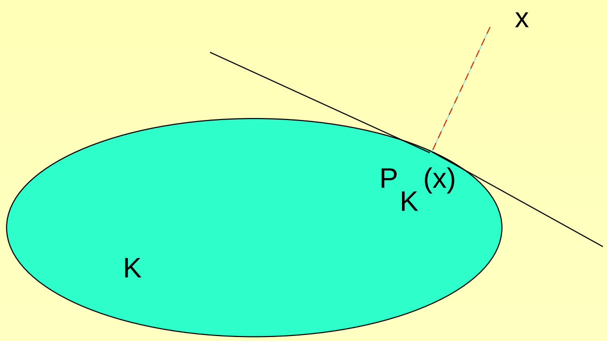


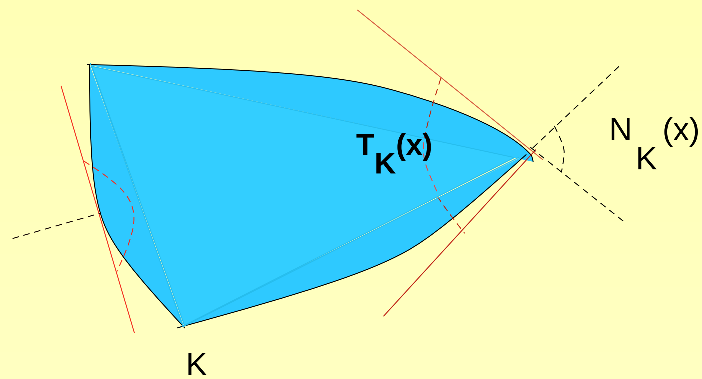
In other words, a trajectory of a projected differential equation is always "trapped" in the constraint set $K = \mathbb{R}^2$ and the velocity field along any such trajectory is not continuous.

PrDE and PDS - II

To rigorously define the two notions, we quickly remind the following:
1) the projection of X onto K by $P_K : X \to K$, with
$||P_K(x) - x|| = \inf_{z \in K} ||z - x||$, for all $z \in X$

2) the tangent cone $T_K(x) = \overline{\bigcup_{h>0} \frac{1}{h}(K - x)}$.



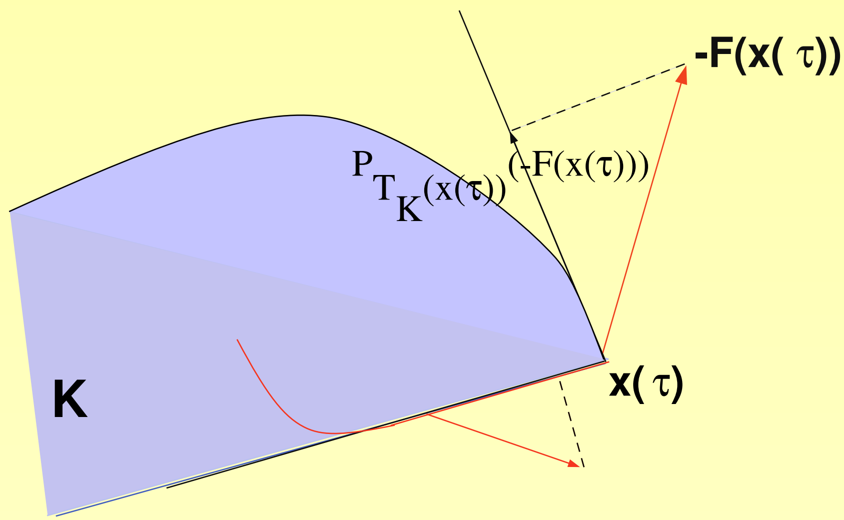


PrDE and PDS - III

Let X, $K \subset X$ and $F : K \rightarrow X$ as before. A PrDE is defined by:

$$\frac{d}{d\tau}(x(\tau)) = P_{T_{K(x(\tau))}}(-F(x(\tau))),\ x(0) \in K,$$

as illustrated next


-F(x(τ))
P
T
K
(x(τ))
(-F(x(τ)))
x(τ)
K

The righthand side of any PrDE is nonlinear and discontinuous.

An existence and uniqueness result was obtained for such equations by [Nagurney '93], for $X := \mathbb{R}^n$, and by [Cojocaru '02] for general Hilbert spaces.

THEOREM: A PrDE

$$\frac{d}{d\tau}(x(\tau)) = P_{T_K(x(\tau))}(-F(x(\tau))),\ x(0) \in K$$

has solutions in $AC([0,\infty), K)$ and are unique through each initial point in K.

A projected dynamical system (PDS) is the dynamical system given by the set of trajectories of a PrDE.

Equilibria of PDS and Variational Inequalities

An equilibrium for a PDS is a point $x \in K$ such that $P_{T_K(x)}(-F(x)) = 0$.

Important feature of any PDS: is intimately related to a variational inequality problem (VI).

VI reminder: Given a Banach space E, $f : K \to E$, and $\ll \cdot, \cdot \gg$ the duality map on E, the variational inequality defined by f and K is

$$VI(f, K) := \begin{cases} \text{find } x_0 \in K \text{ such that} \\ \ll y - x_0, f(x_0) \gg \geq 0, \text{ for all } y \in K \end{cases}$$

Starting point of VI theory: 1964 (Stampacchia); it is now part of calculus of variations; it is used to show existence of equilibria in economic problems & free boundary problems, among others.

The following relation between a PDS and a VI was shown by [Nagurney '93] for $X := \mathbb{R}^n$ and [Cojocaru '02] for any Hilbert space:

THEOREM: The equilibria of a PDS:

$$\frac{d}{d\tau}(x(\tau)) = P_{T_K(x(\tau))}(-F(x(\tau)))$$

are solutions to the VI(F,K):

find $x \in K$ such that $\ll F(x), y - x \gg \geq 0,$ for all $y \in K$

and viceversa.

Evolutionary Variational Inequalities (EVI)

They are a particular case of VI. The EVI we use here are characterized by constraint sets K given as follows:
$K \subset L^p([0,T],\mathbb{R}^q)$ convex, closed, bounded given by:

$$K = \bigcup_{t\in[0,T]} \Big\{ u \in L^p([0,T],\mathbb{R}^q) \,|\, \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\},$$

$$j \in \{1,..,l\} \Big\}.$$

Let $\ll \cdot, \cdot \gg$ be the duality map between $L^p([0,T],\mathbb{R}^q)$ and $(L^p([0,T],\mathbb{R}^q))^*$ given by: $\ll \phi, u \gg := \int_0^T \langle \phi(\tau), u(\tau) \rangle d\tau$, with $\phi \in (L^p([0,T],\mathbb{R}^q))^*$ and $u \in L^p([0,T],\mathbb{R}^q)$.

If we consider a mapping $F : K \to L^p([0,T],\mathbb{R}^q)^*$, then an EVI is the problem

$$\text{find } u \in K \text{ such that } \ll F(u), v-u \gg \geq 0, \ \forall v \in K.$$

[Daniele,'03] showed the following existence theorem for EVI.
THEOREM: If F satisfies either of the following:

- F is hemicontinuous with respect to the strong topology on K, and there exist $A \subseteq K$ nonempty, compact, and $B \subseteq K$ compact such that, for every $v \in K \setminus A$, there exists $u \in B$ with $\ll F(v), u - v \gg \geq 0$;

- F is hemicontinuous with respect to the weak topology on K ;

- F is pseudomonotone and hemicontinuous along line segments,

then the EVI admits a solution over the constraint set K. If F is in addition strictly monotone, then the solution to the EVI is unique.

How PDS and EVI mesh

The original work presented from now on can be found in

[Cojocaru, Daniele, Nagurney,'04] *Projected Dynamical Systems and Evolutionary (Time-dependent) Variational Inequalities on Hilbert Spaces with Applications*, to appear in JOTA;

Online source: http://www.uoguelph.ca/m̃cojocar or

Virtual Center for Supernetworks

http://supernet.som.umass.edu/articles/evipds.pdf

We consider $p := 2$, the Hilbert space $L^2([0,T],\mathbb{R}^q)$ and the EVI

$$\text{find } u \in K \text{ such that } \ll F(u), v-u \gg \geq 0, \ \forall v \in K.$$

where

$$K = \bigcup_{t\in[0,T]} \Big\{ u \in L^2([0,T],\mathbb{R}^q) \,|\, \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T];$$
$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\}$$
$$j \in \{1,..,l\} \Big\},$$

$F : K \to L^2([0,T],\mathbb{R}^q)$ Lipschitz continuous and pseudomonotone.

Let t be arbitrarily fixed in $[0, T]$ and consider the PDS given by:

$$\frac{du(t,\tau)}{dt} = P_{T_{K(u(t,\tau))}}(-F(u(t,\tau))),\ u(t,0) = u(t) \in K,$$

where τ is the evolution time of the PDS and t is the evolution time of the EVI. Note that:

- at each fixed $t \in [0, T]$
  - ⋆ the solution(s) of the EVI represent one or more equilibria of the PDS;
  - ⋆ $\tau \in [0, l]$ is the time it takes the system to reach one of the equilibria on the curve(s);

- as t varies over $[0, T]$, these equilibria describe one (or more) curve(s).

Application to traffic networks

To use these concepts we perform the following steps:

- we discretize the evolution time interval $[0, T]$ of the EVI;
- we obtain a finite collection of PDS's, defined on distinct closed, convex sets K_t;
- we compute the equilibria of each PDS, i.e., find the equilibria at the chosen $t \in [0, T]$;
- we interpolate the sequence of equilibria and obtain an approximation of the curve(s) of equilibria.

- Let $q := 2, T := 2$ and a network consisting of a single origin/destination pair of nodes with two connecting paths of a single link each;

$$K := \bigcup_{t \in [0,2]} \Big\{ u \in L^2([0,2], \mathbb{R}^2) \mid (0,0) \leq (u_1(t), u_2(t)) \leq \left(t, \frac{3}{2}t\right),$$
$$u_1(t) + u_2(t) = t, \text{ a.e. on } [0,2] \Big\}.$$

- Cost functions: $2u_1(t) - 1.5$ on path 1 and $u_2(t) - 1$ on path 2;
- Let $F : L^2([0,2], \mathbb{R}^2) \to L^2([0,2], \mathbb{R}^2)$,
  $F(u_1(t), u_2(t)) = (2u_1(t) - 1.5, u_2(t) - 1)$;
- EVI theory $\implies$ unique equilibrium because F is strictly monotone; evidently, F is Lipschitz.

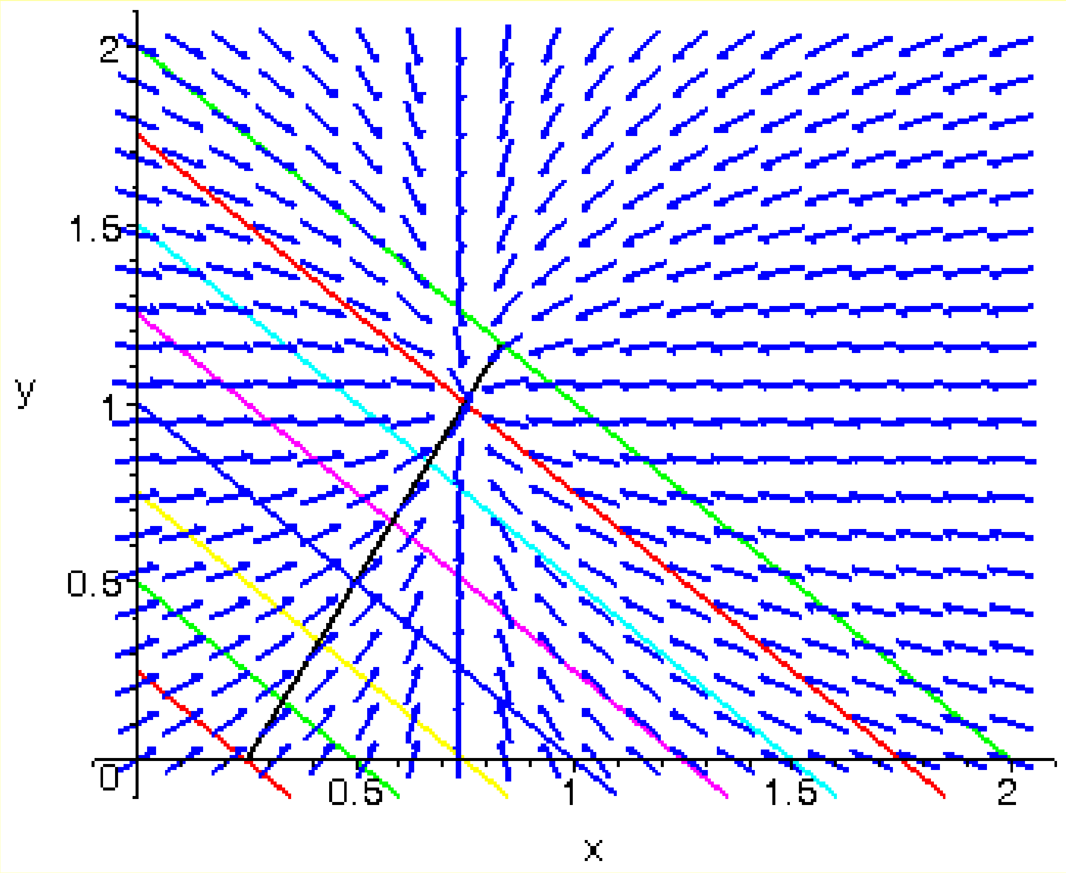
- Choose

$$t_0 \in \left\{ \frac{k}{4} \mid k \in \{0, .., 8\} \right\}.$$

- For t_0 as above we get a sequence of sets $K_{t_0} := \{\{[0, t_0] \times [0, \frac{3}{2}t_0]\} \cap \{u_1 + u_2 = t_0\}\}$;

- For each such set we compute the unique equilibrium at t_0: $\{(0,0), (\frac{1}{4}, 0), (\frac{1}{3}, \frac{1}{6}), (\frac{5}{12}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}), (\frac{7}{12}, \frac{2}{3}), (\frac{2}{3}, \frac{5}{6}), (\frac{3}{4}, 1), (\frac{5}{6}, \frac{7}{6})\}$.

- Interpolating we obtain the following figure:



where $x := u_1$ and $y := u_2$.

Final remarks:

- The PDS theory gives the natural environment in which an EVI can be applied;
- The PDS-EVI mesh opens more questions to study, some of them theoretical and some regarding the future PDS-EVI applications;
- As far as applications, the PDS-EVI has perhaps the strongest potential to model applied problems involving layered time-frames, as our application shows.