Dynamic Electric Power Supply Chains and Transportation Networks: an Evolutionary Variational Inequality Formulation

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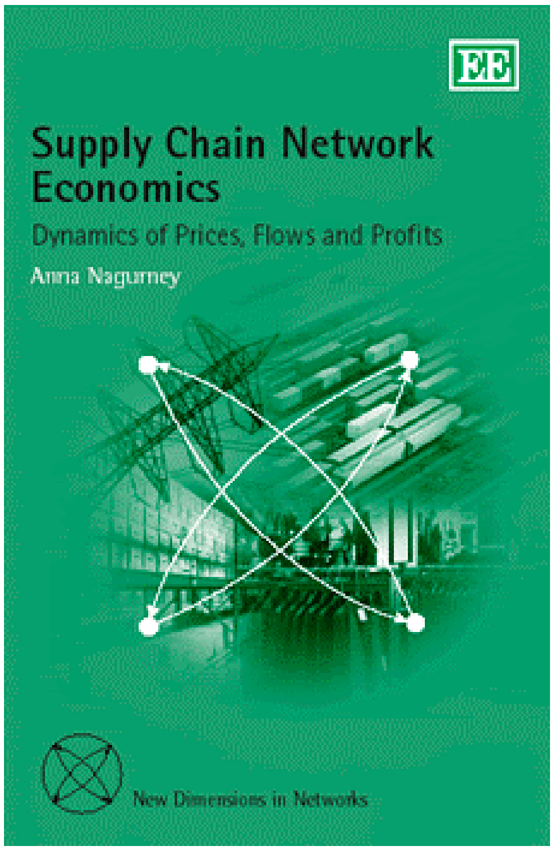
The New Book



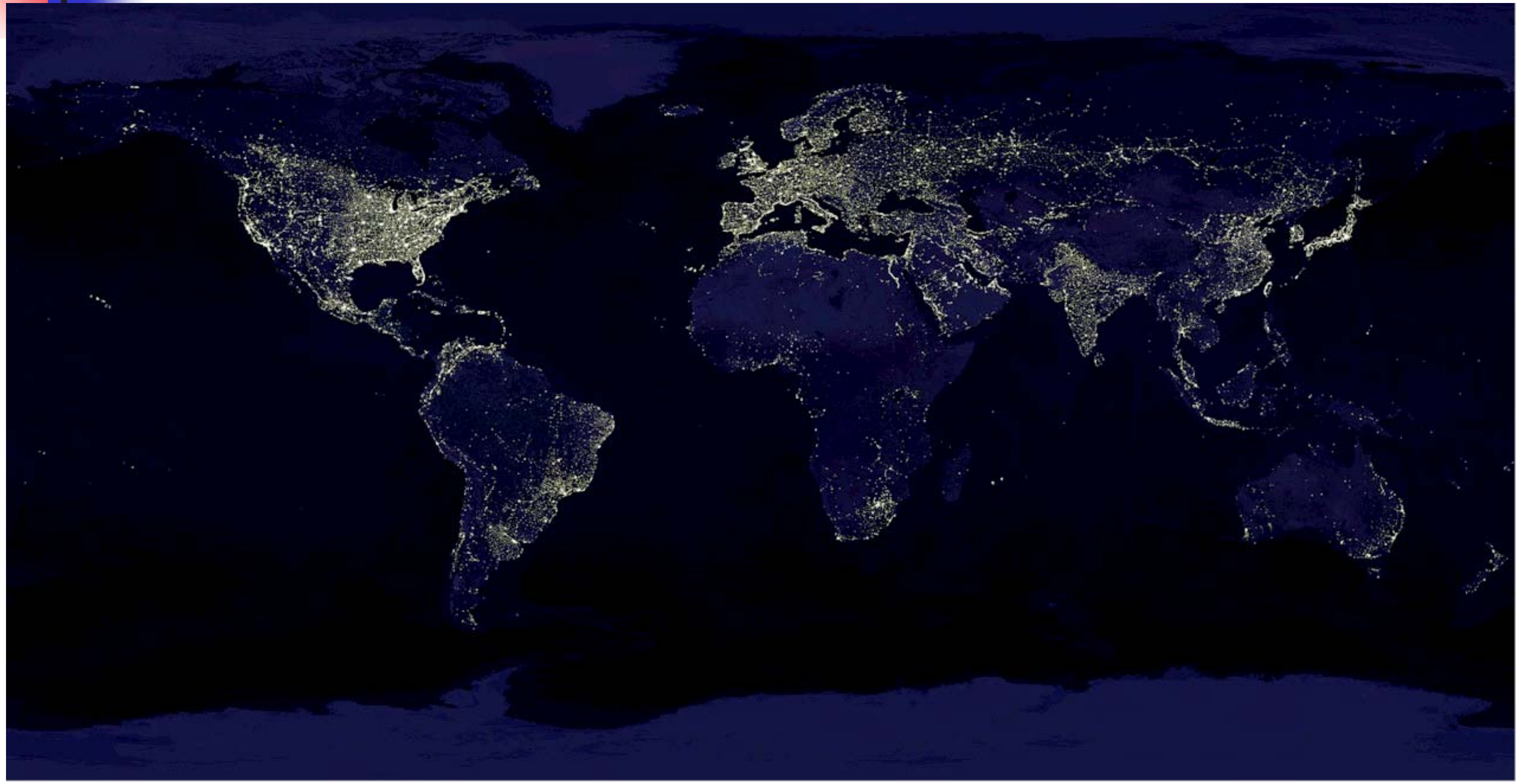
- ***Supply Chain Network Economics* (New Dimensions in Networks)**

  Anna Nagurney

  Edward Elgar Publishing Inc.

- **Available July 2006!**

Electricity is Modernity



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Motivation

- In US: half a trillion dollars worth of net assets
- Consumes almost 40% of domestic primary energy
- Electric power supply chains, provide the foundations for the functioning of our modern economies and societies.
  - Communication, transportation, heating, lighting, cooling, computers and electronics.
  - August 14, 2003, blackout in the Midwest, the Northeastern United States, and Ontario, Canada.
  - Two significant power outages during the month of September 2003 – one in England and one in Switzerland and Italy.
- Deregulation: from vertically integrated to competitive markets
  - In US, Europe and many other countries
- Inelastic, seasonal demand.

Literature

- Kahn (1998)
- Day et al. (2002)
- Schweppe et al. (1988)
- Hogan (1992)
- Chao and Peck (1996)
- Wu et al. (1996)
- Willems (2002)
- Casazza and Delea (2003)
- Zaccour (1998) and Singh (1999).

Objectives

- The objective of this research was to develop a dynamic electric power supply chain network equilibrium model with exogenous time-varying demand

- The theory that has originated from the study of transportation networks was utilized to construct this time-dependent equilibrium modeling framework for electric power supply chain networks

- The new dynamic electric power supply chain network model that we developed in this research is also motivated by the unification of projected dynamical systems theory and evolutionary (infinite-dimensional) variational inequalities

Outline

- The static electric power network model with fixed demands
- The supernetwork equivalence of the electric power supply chain networks and the transportation networks
  - Overview of the transportation network equilibrium models
  - The supernetwork equivalence of the transportation networks and the electric power supply chain networks with fixed demands
- The electric power supply chain network model with time-varying demands
  - Evolutionary variational inequalities and projected dynamical systems; Applications to transportation network equilibrium
  - The computation of the electric power supply chain network equilibrium model with time-varying demands.

Some of the Related Literature

- Beckmann, M. J., McGuire, C. B., and Winsten, C. B. (1956), Studies in the Economics of Transportation. Yale University Press, New Haven, Connecticut.

- Nagurney, A (1999), Network Economics: A Variational Inequality Approach, Second and Revised Edition, Kluwer Academic Publishers, Dordrecht, The Netherlands.

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- Nagurney, A (2005), On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations, Appears in *Transportation Research E* 42: (2006) pp 293-316.

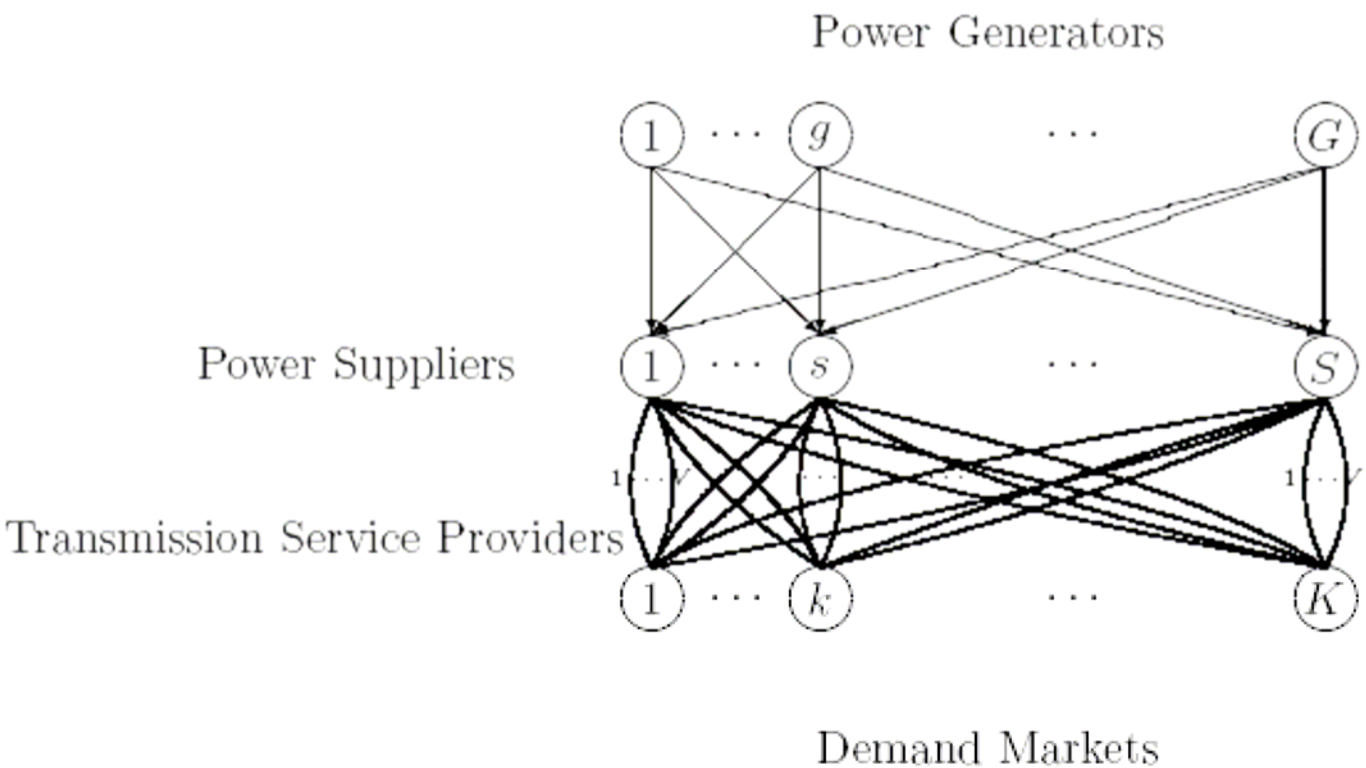
Some of the Related Literature (Cont'd)

- Nagurney, A. and Matsypura, D. (2004), A Supply Chain Network Perspective for Electric Power Generation, Supply, Transmission, and Consumption, *Proceedings of the International Conference on Computing, Communications and Control Technologies*, Austin, Texas, Volume VI: (2004) pp 127-134.

- Nagurney, A. and Liu, Z (2005), Transportation Network Equilibrium Reformulations of Electric Power Networks with Computations.

- Wu, K., Nagurney, A., Liu, Z. and Stranlund, J. (2006), Modeling Generator Power Plant Portfolios and Pollution Taxes in Electric Power Supply Chain Networks: A Transportation Network Equilibrium Transformation, *Transportation Research D* 11: (2006) pp 171-190.

The Electric Power Supply Chain Network Equilibrium Model with Fixed Demands



The Behavior of Power Generator and Their Optimality Conditions

- Conservation of flow equations must hold for each generator

$$\sum_{s=1}^{S} q_{gs} = q_g, \quad g = 1, \ldots, G. \qquad (1)$$

- Power generator's optimization problem

$$\text{Maximize} \quad \sum_{s=1}^{S} \rho_{1gs}^{*} q_{gs} - f_g(Q^1) - \sum_{s=1}^{S} c_{gs}(q_{gs})$$

subject to:

$$q_{gs} \geq 0, \qquad s = 1, \ldots, S.$$

- The optimality conditions of the generators

$$\sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial f_g(Q^{1*})}{\partial q_{gs}} + \frac{\partial c_{gs}(q_{gs}^{*})}{\partial q_{gs}} - \rho_{1gs}^{*} \right] \times [q_{gs} - q_{gs}^{*}] \geq 0, \quad \forall Q^1 \in R_{+}^{GS}.$$

The Behavior of Power Suppliers

- Supplier's optimization problem

$$\text{Maximize} \quad \sum_{k=1}^{K}\sum_{v=1}^{V}\rho_{2sk}^{v*}q_{sk}^{v} - c_s(Q^1) - \sum_{g=1}^{G}\rho_{1gs}^{*}q_{gs} - \sum_{g=1}^{G}\hat{c}_{gs}(q_{gs}) - \sum_{k=1}^{K}\sum_{v=1}^{V}c_{sk}^{v}(q_{sk}^{v})$$

subject to:

$$\sum_{k=1}^{K}\sum_{v=1}^{V}q_{sk}^{v} = \sum_{g=1}^{G}q_{gs}$$

$$q_{gs} \geq 0, \quad g = 1, \ldots, G, \tag{8}$$

$$q_{sk}^{v} \geq 0, \quad k = 1, \ldots, K; v = 1, \ldots, V.$$

- For notational convenience, we let

$$h_s \equiv \sum_{g=1}^{G} q_{gs}, \quad s = 1, \ldots, S. \tag{13}$$

The Optimality Conditions of the Power Suppliers

- The optimality conditions of the suppliers

$$\sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{t=1}^{T} \left[\frac{\partial c_{sk}^t(q_{sk}^{t*})}{\partial q_{sk}^t} - \rho_{2sk}^{t*} \right] \times [q_{sk}^t - q_{sk}^{t*}]$$

$$+ \sum_{g=1}^{G} \sum_{s=1}^{S} \left[\frac{\partial \hat{c}_{gs}(q_{gs}^*)}{\partial q_{gs}} + \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*] \geq 0, \qquad \forall (Q^1, Q^2, h) \in \mathcal{K}^3,$$

where $\mathcal{K}^3 \equiv \{(h, Q^2, Q^1) | (h, Q^2, Q^1) \in R_+^{S(1+TK+G)}$ and (8) and (13) hold$\}$.

The Equilibrium Conditions at the Demand Markets

- Conservation of flow equations must hold

$$d_k = \sum_{s=1}^{S} \sum_{t=1}^{T} q_{sk}^t, \quad k = 1, \ldots, K. \tag{17}$$

- The vector (Q^{2*}, $\rho_3{}^*$) is an equilibrium vector if for each *s, k, v* combination:

$$\rho_{2sk}^{t*} + \hat{c}_{sk}^t(Q^{2*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{sk}^{t*} > 0, \\ \geq \rho_{3k}^*, & \text{if} \quad q_{sk}^{t*} = 0, \end{cases}$$

Electric Power Supply Chain Network Equilibrium (For Fixed Demands at the Markets)

- **<u>Definition</u>**: The equilibrium state of the electric power supply chain network is one where the electric power flows between the tiers of the network coincide and the electric power flows satisfy the sum of the optimality conditions of the power generators and the suppliers, and the equilibrium conditions at the demand markets.

Variational Inequality Formulation

- Determine $(q^*, h^*, Q^{1*}, Q^{2*}) \in \mathcal{K}^5$ satisfying

$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{g=1}^{G}\sum_{s=1}^{S} \left[\frac{\partial c_{gs}(q_{gs}^*)}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(q_{gs}^*)}{\partial q_{gs}}\right] \times [q_{gs} - q_{gs}^*]$$

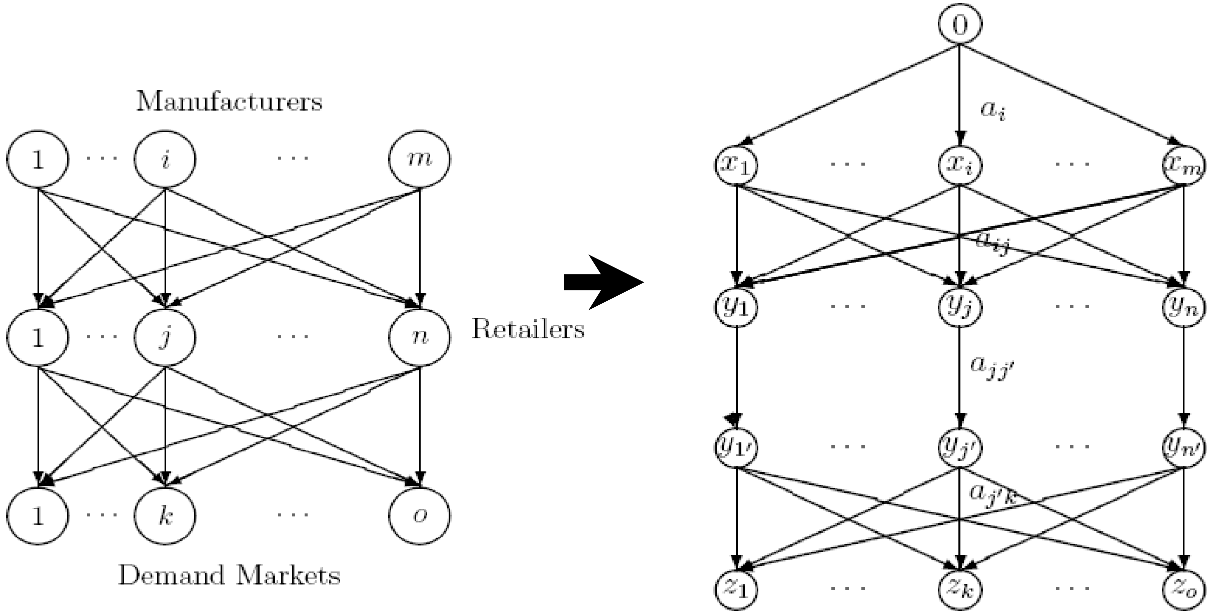
$$+\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{v=1}^{V} \left[\frac{\partial c_{sk}^v(q_{sk}^{v*})}{\partial q_{sk}^v} + \hat{c}_{sk}^v(Q^{2*})\right] \times [q_{sk}^v - q_{sk}^{v*}] \geq 0, \quad \forall (q, h, Q^1, Q^2) \in \mathcal{K}^5, \qquad (20)$$

$$where\ \mathcal{K}^5 \equiv \{(q, h, Q^1, Q^2) | (q, h, Q^1, Q^2) \in R_+^{G+S+GS+VSK}$$

$$and\ (1),\ (8),\ (13),\ and\ (17)\ hold\}.$$

The Supernetwork Equivalence of Supply Chain Network Equilibrium and Transportation Network Equilibrium

- Nagurney, A. (2006), On the Relationship Between Supply Chain and Transportation Network Equilibria: A Supernetwork Equivalence with Computations, *Transportation Research E (2006)* 42: (2006) pp 293-316



Overview of the Transportation Network Equilibrium Model with Fixed Demands

- Smith, M. J. (1979), Existence, uniqueness, and stability of traffic equilibria. *Transportation Research 13B*, 259-304.

- Dafermos, S. (1980), Traffic equilibrium and variational inequalities. *Transportation Science* 14, 42-54.

- In equilibrium, the following conditions must hold for each O/D pair and each path.

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0. \end{cases} \qquad (27)$$

- A path flow pattern is a transportation network equilibrium if and only if it satisfies the variational inequality:

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^* \right] \geq 0, \quad \forall x \in \mathcal{K}^6.$$

The Supernetwork Equivalence of the Electric Power Network and the Transportation Network

- The fifth chapter of the Beckmann, McGuire, and Winsten's classic book, "Studies in the Economics of Transportation" (1956), described some "unsolved problems" including a single commodity network equilibrium problem that the authors intuited could be generalized to capture electric power networks.
- We took up this challenge of establishing the relationship and application of transportation network equilibrium models to electric power networks.

Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands

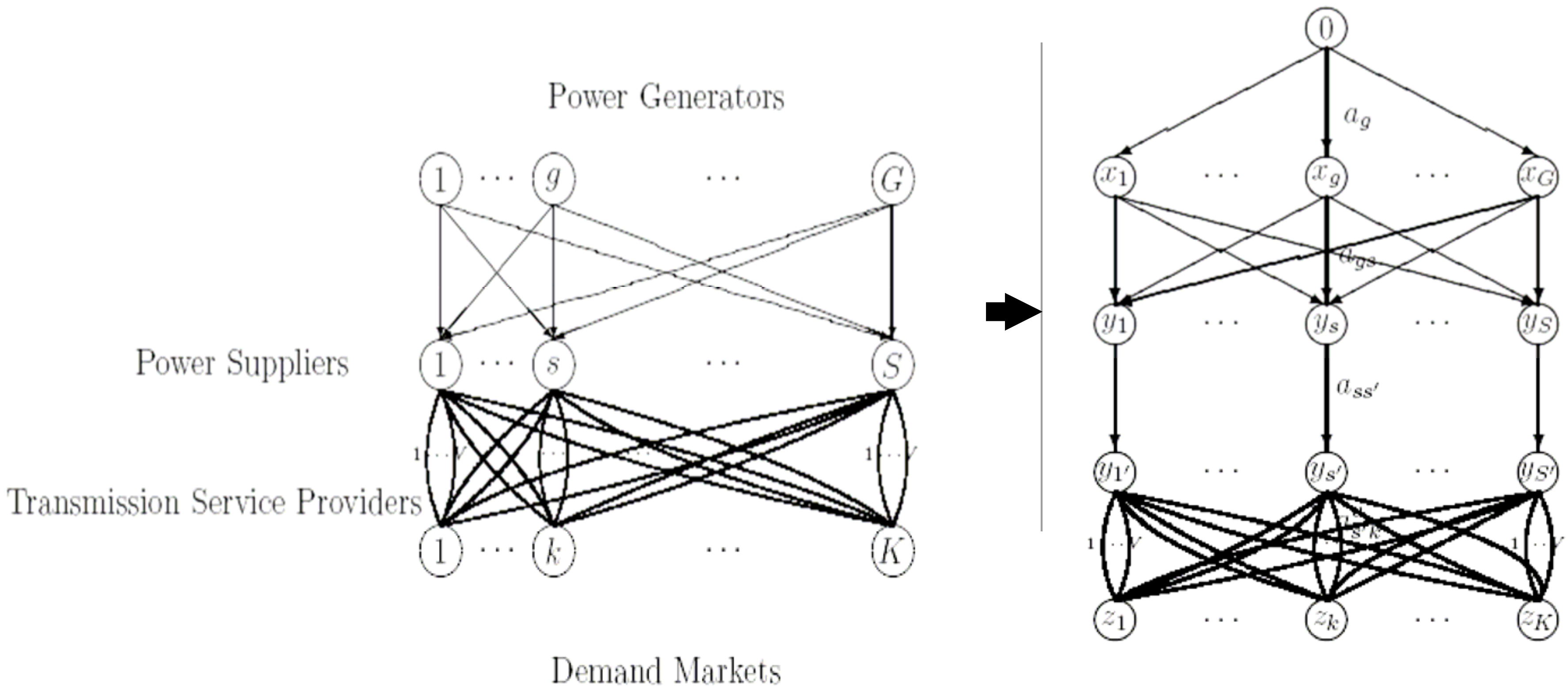


Figure 3

Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands

- The following conservation of flow equations must hold on the equivalent transportation network:

$$f_{a_g} = \sum_{s=1}^{S} \sum_{s'=1}^{S'} \sum_{k=1}^{K} \sum_{v=1}^{V} x_{p^v_{gss'k}}, \quad g = 1, \ldots, G, \tag{30}$$

$$f_{a_{gs}} = \sum_{s'=1}^{S'} \sum_{k=1}^{K} \sum_{v=1}^{V} x_{p^v_{gss'k}}, \quad g = 1, \ldots, G; s = 1, \ldots, S,$$

$$f_{a_{ss'}} = \sum_{g=1}^{G} \sum_{k=1}^{K} \sum_{v=1}^{V} x_{p^v_{gss'k}}, \quad ss' = 11', \ldots, SS',$$

$$f_{a^v_{s'k}} = \sum_{g=1}^{G} \sum_{s=1}^{S} x_{p^v_{gss'k}}, \quad s' = 1, \ldots, S'; v = 1, \ldots, V; k = 1, \ldots, K.$$

$$d_{w_k} = \sum_{g=1}^{G} \sum_{s=1}^{S} \sum_{s'=1}^{S'} \sum_{v=1}^{V} x_{p^v_{gss'k}}, \quad k = 1, \ldots, K. \tag{34}$$

Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands

- We can construct a feasible link flow pattern for the equivalent transportation network based on the corresponding feasible electric power flow pattern in the electric power supply chain network model in the following way:

$$q_g \equiv f_{a_g}, \quad g = 1, \ldots, G, \tag{35}$$

$$q_{gs} \equiv f_{a_{gs}}, \quad g = 1, \ldots, G; s = 1, \ldots, S, \tag{36}$$

$$h_s \equiv f_{a_{ss'}}, \quad ss' = 11', \ldots, SS', \tag{37}$$

$$q^v_{sk} = f_{a^v_{s'k}}, \quad s = s' = 1', \ldots, S'; v = 1, \ldots, V; k = 1, \ldots, K, \tag{38}$$

$$d_k = \sum_{s=1}^{S} \sum_{v=1}^{V} q^v_{sk}, \quad k = 1, \ldots, K. \tag{39}$$

Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands

- We assign user (travel) costs on the links of the transportation network as follows:

$$c_{a_g} \equiv \frac{\partial f_g}{\partial q_g}, \quad g = 1, \ldots, G, \tag{40}$$

$$c_{a_{gs}} \equiv \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}}, \quad g = 1, \ldots, G; s = 1, \ldots, S, \tag{41}$$

$$c_{a_{ss'}} \equiv \frac{\partial c_s}{\partial h_s}, \quad ss' = 11', \ldots, SS'. \tag{42}$$

$$c_{a^v_{s'k}} \equiv \frac{\partial c^v_{sk}}{\partial q^v_{sk}} + \hat{c}^v_{sk}, \quad s' = s = 1, \ldots, S; v = 1, \ldots, V; k = 1, \ldots, K. \tag{43}$$

Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands

- Path cost

$$C_{p^v_{gss'k}} = c_{a_g} + c_{a_{gs}} + c_{a_{ss'}} + c_{a^v_{s'k}} = \frac{\partial f_g}{\partial q_g} + \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}} + \frac{\partial c_s}{\partial h_s} + \frac{\partial c^v_{sk}}{\partial q^v_{sk}} + \hat{c}^v_{sk}. \quad (44)$$

- We assign the (travel) demands associated with the O/D pairs as follows:

$$d_{w_k} \equiv d_k, \quad k = 1, \ldots, K,$$

- The (travel) disutilities:

$$\lambda_{w_k} \equiv \rho_{3k}, \quad k = 1, \ldots, K.$$

- Substitute into the equilibrium condition (27)

$$C_{p^v_{gss'k}} - \lambda^*_{w_k} = \frac{\partial f_g}{\partial q_g} + \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}} + \frac{\partial c_s}{\partial h_s} + \frac{\partial c^v_{sk}}{\partial q^v_{sk}} + \hat{c}^v_{sk} - \rho^*_{3k} \begin{cases} = 0, & \text{if} \quad x^*_{p^v_{gss'k}} > 0, \\ \geq 0, & \text{if} \quad x^*_{p^v_{gss'k}} = 0. \end{cases}$$

Transportation Network Equilibrium Reformulation of the Electric Power Network Model with Fixed Demands

Theorem

A solution $(q^*, h^*, Q^{1*}, Q^{2*}) \in \mathcal{K}^5$ of the variational inequality (20) governing the electric power supply chain network equilibrium coincides with (via (35) – (43)) the feasible link flow pattern for the supernetwork $\mathcal{G}_S$ constructed above and satisfies variational inequality (20). Hence, it is a transportation network equilibrium

Theorem

A path flow pattern on the supernetwork in Figure 3 is a transportation network equilibrium if and only if it satisfies the variational inequality problem: determine $x^*_{p^v_{gss'k}} \geq 0$, for all g, s, s', v, k and satisfying (34) such that

$$\sum_{g=1}^{G}\sum_{s=1}^{S}\sum_{s'=1'}^{S'}\sum_{v=1}^{V}\sum_{k=1}^{K} C_{p^v_{gss'k}}(x^*) \times \left[x_{p^v_{gss'k}} - x^*_{p^v_{gss'k}}\right] \geq 0,$$

$$\forall x_{p^v_{gss'k}} \geq 0, \forall g, s, s', v, k \text{ and satisfying} (34). \qquad (50)$$

Finite-Dimentional Variational Inequalities and Projected Dynamical Systems Literature

- Dupuis, P., Nagurney, A., (1993). Dynamical systems and variational inequalities. *Annals of Operations Research* 44, 9-42.

- Nagurney, A., Zhang, D., (1996). Projected Dynamical Systems and Variational Inequalities with Applications. Kluwer Academic Publishers, Boston, Massachusetts.

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More Finite-Dimentional Variational Inequalities Literature

- Smith, M. J. (1979), Existence, uniqueness, and stability of traffic equilibria. *Transportation Research 13B*, 259-304.
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The Evolutionary Variational Inequalities and Projected Dynamical Systems Literature

- Cojocaru, M.-G., Jonker, L. B., (2004). Existence of solutions to projected differential equations in Hilbert spaces. *Proceedings of the American Mathematical Society* 132, 183–193.

- Cojocaru, M.-G., Daniele, P., Nagurney, A., (2005a). Projected dynamical systems and evolutionary variational inequalities via Hilbert spaces with applications. *Journal of Optimization Theory and Applications* 27, no. 3, 1-15.

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More Evolutionary Variational Inequalities and Projected Dynamical Systems Literature

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Finite-Dimensional Projected Dynamical Systems

- Finite-Dimensional Projected Dynamical Systems (PDSs) (Dupuis and Nagurney (1993))
  - PDS_t describes how the state of the network system approaches an equilibrium point on the curve of equilibria at time t.
  - For almost every moment ‘t’ on the equilibria curve, there is a PDS_t associated with it.
  - A PDS_t is usually applied to study small scale time dynamics, i.e $[t, t+\tau]$

Finite-Dimensional Projected Dynamical Systems

Definition: $$\frac{dx(t)}{dt} = \Pi_{\mathcal{K}}(x(t), -F(x(t))).$$

In this formulation, $\mathcal{K}$ is a convex polyhedral set in R^n, $F : \mathcal{K} \to R^n$ is a Lipschitz continuous function with linear growth and $\Pi_{\mathcal{K}} : R \times \mathcal{K} \to R^n$ is the Gateaux directional derivative

$$\Pi_{\mathcal{K}}(x, -F(x)) = \lim_{\delta \to 0^+} \frac{P_{\mathcal{K}}(x - \delta F(x)) - x}{\delta}$$

of the projection operator $P_{\mathcal{K}} : R^n \to \mathcal{K}$, given by

$$||P_{\mathcal{K}}(z) - z|| = \inf_{y \in \mathcal{K}} ||y - z||$$

Projected Dynamical Systems and Finite-Dimensional Variational Inequalities

Theorem

The equilibria of a PDS:

$$\frac{\partial x(t)}{\partial t} = \Pi_{\mathcal{K}}(x(t), -F(x(t)))$$

that is, $x^* \in \mathcal{K}$ *such that*

$$\Pi_{\mathcal{K}}(x^*, -F(x^*)) = 0$$

are solutions to the $VI(F, K)$*: find* $x^* \in \mathcal{K}$ *such that*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{K}$$

and vice versa.

Infinite-Dimensional Projected Dynamical Systems

Definition: $$\frac{dx(t,\tau)}{d\tau} = \Pi_{\hat{\mathcal{K}}}(x(t,\tau), -F(x(t,\tau))), \quad x(t,0) \in \hat{\mathcal{K}}, \tag{54}$$

where $$\Pi_{\hat{\mathcal{K}}}(y, -F(y)) = \lim_{\delta \to 0^+} \frac{P_{\hat{\mathcal{K}}}((y - \delta F(y)) - y)}{\delta}, \quad \forall y \in \hat{\mathcal{K}}$$

with the projection operator $P_{\hat{\mathcal{K}}} : H \to \hat{\mathcal{K}}$ given by

$$\|P_{\hat{\mathcal{K}}}(z) - z\| = \inf_{y \in \hat{\mathcal{K}}} \|y - z\|,$$

The feasible set $\hat{\mathcal{K}}$ is defined as follows

$$\hat{\mathcal{K}} = \bigcup_{t \in [0,T]} \Big\{ u \in L^p([0,T], R^q) \,|\, \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\}, j \in \{1,\ldots,l\} \Big\}.$$

Evolutionary Variational Inequalities

- Evolutionary Variational Inequalities (EVIs)
  - EVI provides a curve of equilibria of the network system over a finite time interval [0,T]
  - An EVI is usually used to model large scale time, i.e, [0, T]
  - EVIs have been applied to time-dependent equilibrium problems in transportation, and in economics and finance.

Evolutionary Variational Inequalities

Define $$\langle\langle \Phi, x \rangle\rangle = \int_0^T \langle \Phi(t), x(t) \rangle dt \qquad (52)$$

EVI:

determine $x \in \hat{\mathcal{K}} : \langle\langle F(x), z - x \rangle\rangle \geq 0, \quad \forall z \in \hat{\mathcal{K}}. \qquad (53)$

where $$\hat{\mathcal{K}} = \bigcup_{t\in[0,T]} \Big\{ u \in L^p([0,T], R^q) \,|\, \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T];$$

$$\sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\}, j \in \{1,\ldots,l\} \Big\}.$$

Projected Dynamical Systems and Evolutionary Variational Inequalities

- Cojocaru, Daniele, and Nagurney (2005b) showed the following:

Theorem

Assume that $\hat{\mathcal{K}} \subseteq H$ is non-empty, closed and convex and $F : \hat{\mathcal{K}} \to H$ is a pseudo-monotone Lipschitz continuous vector field, where H is a Hilbert space. Then the solutions of EVI (53) are the same as the critical points of the projected differential equation (54) that is, they are the functions $x \in \hat{\mathcal{K}}$ such that

$$\Pi_{\hat{\mathcal{K}}}(x(t), -F(x(t))) = 0,$$

and vice versa.

Projected Dynamical Systems and Evolutionary Variational Inequalities

The solutions to the evolutionary variational inequality:

$$\text{determine} \quad x \in \hat{\mathcal{K}} : \quad \int_0^T \langle F(x(t)), z(t) - x(t) \rangle dt \geq 0, \quad \forall z \in \hat{\mathcal{K}},$$

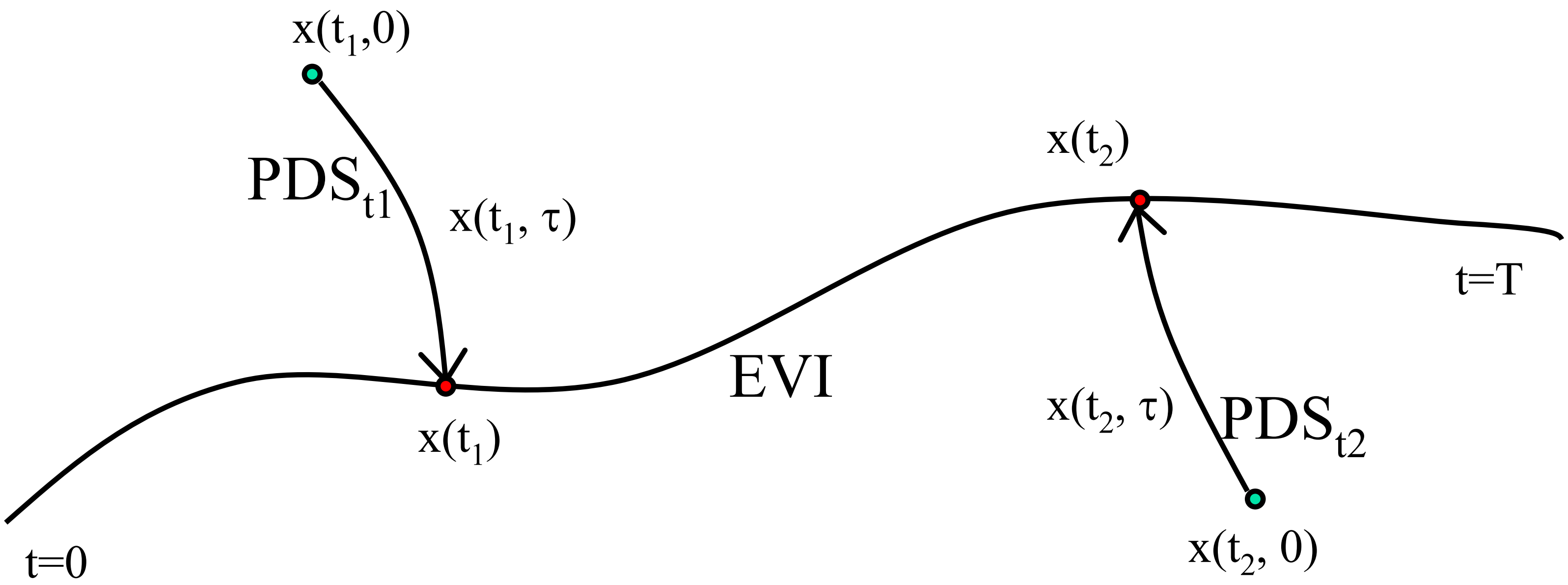
are the same as the critical points of the equation:

$$\frac{dx(t,\tau)}{d\tau} = \Pi_{\hat{\mathcal{K}}}(x(t,\tau), -F(x(t,\tau))),$$

that is, the points such that

$$\Pi_{\hat{\mathcal{K}}}(x(t,\tau), -F(x(t,\tau))) \equiv 0 \text{ a.e. in } [0,T],$$

A Pictorial of EVIs and PDSs



The EVI Formulation of the Transportation Network Model with Time-Varying Demands

<u>Define</u> $\langle\langle \Phi, x \rangle\rangle = \int_0^T \langle \Phi(t), x(t) \rangle dt$

EVI Formulation:

$$\text{determine} \quad x \in \hat{\mathcal{K}} : \langle\langle C(x), z - x \rangle\rangle \geq 0, \quad \forall z \in \hat{\mathcal{K}}, \qquad (61)$$

where C is the vector of path costs.

<u>Feasible set</u>

$$\hat{\mathcal{K}} = \left\{ x \in L^2([0,T], R^Q) : 0 \leq x(t) \leq \mu \text{ a.e. in } [0,T]; \sum_{p \in P_w} x_p(t) = d_w(t), \forall w, \text{ a.e. in } [0,T] \right\}.$$

The Numerical Solution of Evolutionary Variational Inequalities
(Cojocaru, Daniele, and Nagurney (2005 a, b, c))

- The vector field *F* satisfies the requirement in the preceding Theorem.
- We first discretize time horizon T. (Barbagallo, A., (2005))
- At each fixed time point, we solve the associated finite dimensional projected dynamical system PDS_t
- We use the Euler method to solve the finite dimensional projected dynamical system PDS_t.

The Euler Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$ and set $T = 0$. T is an iteration counter which may also be interpreted as a time period.

Step 1: Computation

Compute X^{T+1} by solving the variational inequality problem:

$$X^{T+1} = P_{\mathcal{K}}(X^T - \alpha_T F(X^T)),$$

where $\{\alpha_T\}$ is a sequence of positive scalers satisfying: $\sum_{T=0}^{\infty} \alpha_T = \infty$, $\alpha_T \to 0$ as $T \to \infty$. and $P_{\mathcal{K}}$ is the projection of X on the set $\mathcal{K}$ defined as:

$$y = P_{\mathcal{K}} X = \arg\min_{z \in \mathcal{K}} \| X - z \| .$$

Step 2: Convergence Verification

If $\| X^{T+1} - X^T \| \leq \epsilon$, for some $\epsilon > 0$, a prespecified tolerance, then stop; else, set $T = T + 1$, and go to Step 1.

The EVI Formulation of the Electric Power Network Model with Time-Varying Demands

- We know that the electric power supply chain network equilibrium problem with fixed demands can be reformulated as a fixed demand transportation network equilibrium problem in path flows over the equivalent transportation network.

- Evolutionary variational inequality (61) provides us with a dynamic version of the electric power supply chain network problem in which the demands vary over time, where the path costs are given by (44) but these are functions of path flows that now vary with time.

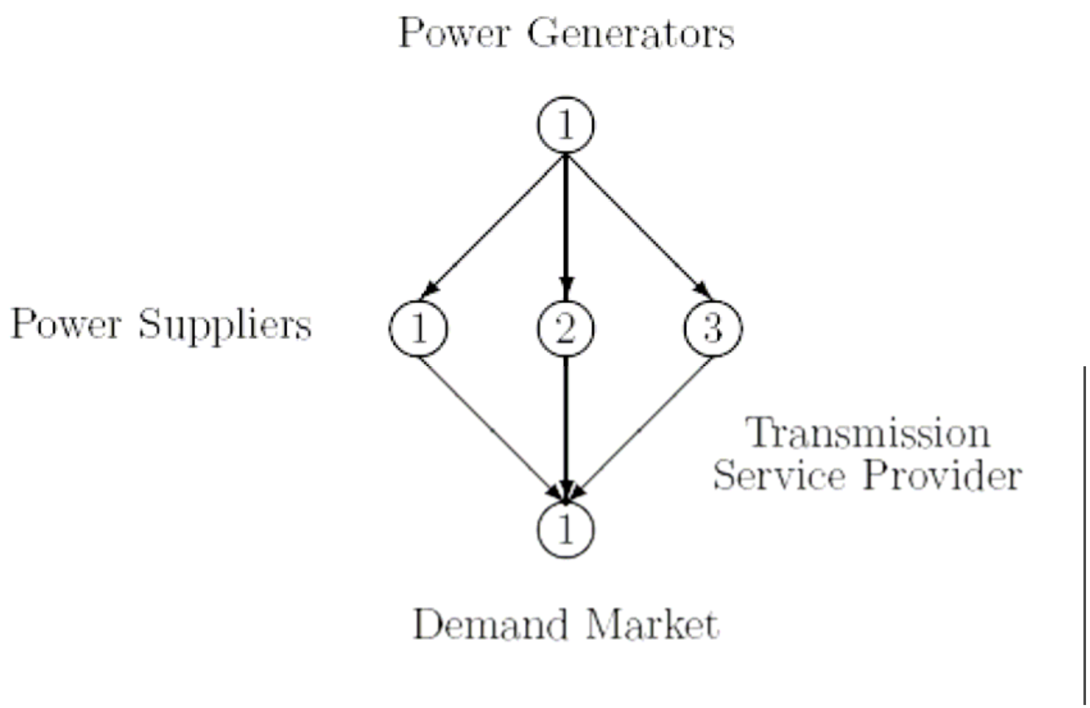
- Evolutionary variational inequality (61) is a dynamic (and infinite-dimensional) version of variational inequality (50) with the path costs defined in (44).

Solving Electric Power Supply Chain Network Model with Time-Varying Demands

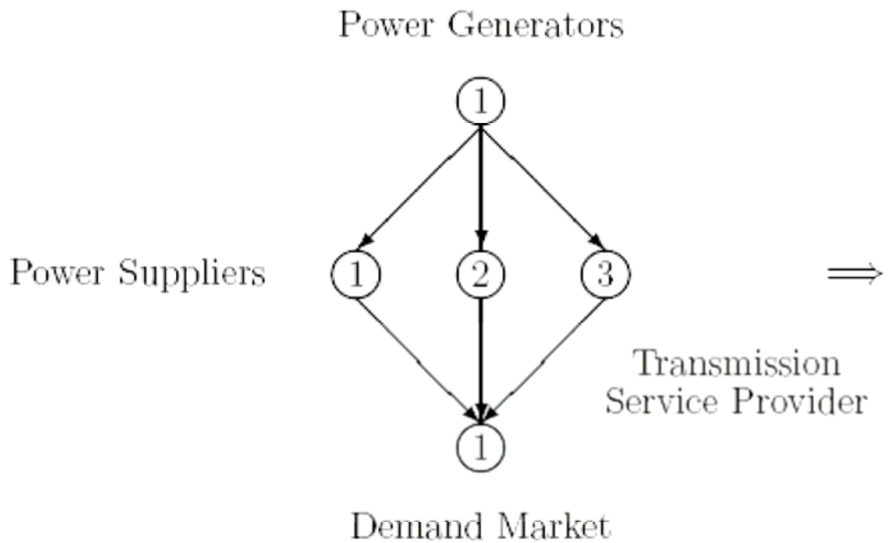
- First, construct the equivalent transportation network equilibrium model
- Solve the transportation network equilibrium model with time-varying demands
- Convert the solution of the transportation network into the time-dependent electric power supply chain network equilibrium model

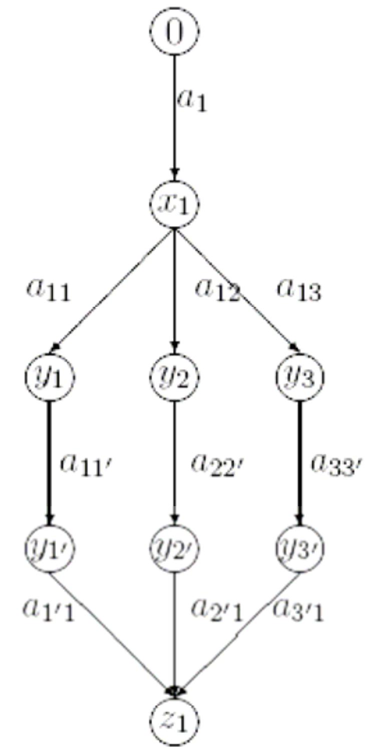
Dynamic Electric Power Supply Chain Network Examples with Computations

- Example 1



Numerical Example 1





Corresponding Supernetwork

Numerical Example 1

- Generating cost functions

$$f_1(q_1(t)) = 2.5(q_1(t))^2 + 2q_1(t).$$

- Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{12}(q_{12}(t)) = .5(q_{12}(t))^2 + 2.5q_{12}(t),$$

$$c_{13}(q_{13}(t)) = .5(q_{13}(t))^2 + 1.5q_{13}(t).$$

Numerical Example 1

- Operating cost functions of the suppliers

$$c_1(Q^1(t)) = .5(q_{11}(t))^2, \quad c_2(Q^1(t)) = .5(q_{12}(t))^2, \quad c_3(Q^1(t)) = .5(q_{13}(t))^2.$$

- Unit transaction cost between the suppliers and the demand markets

$$\hat{c}^1_{11}(Q^2(t)) = q^1_{11}(t) + 1, \quad \hat{c}^1_{21}(Q^2(t)) = q^1_{21}(t) + 5, \quad \hat{c}^1_{31}(Q^2(t)) = q^1_{31}(t) + 10.$$

Numerical Example 1

- Three paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}), \quad p_3 = (a_1, a_{13}, a_{33'}, a_{3'1}).$$

- The time-varying demand function

$$d_{w_1}(t) = d_1(t) = 41 + 10t.$$

The Solution of Numerical Example 1

- Explicit Solution
  - Path flows

$$x^*_{p_1}(t) = 3.33t + 14.78,$$

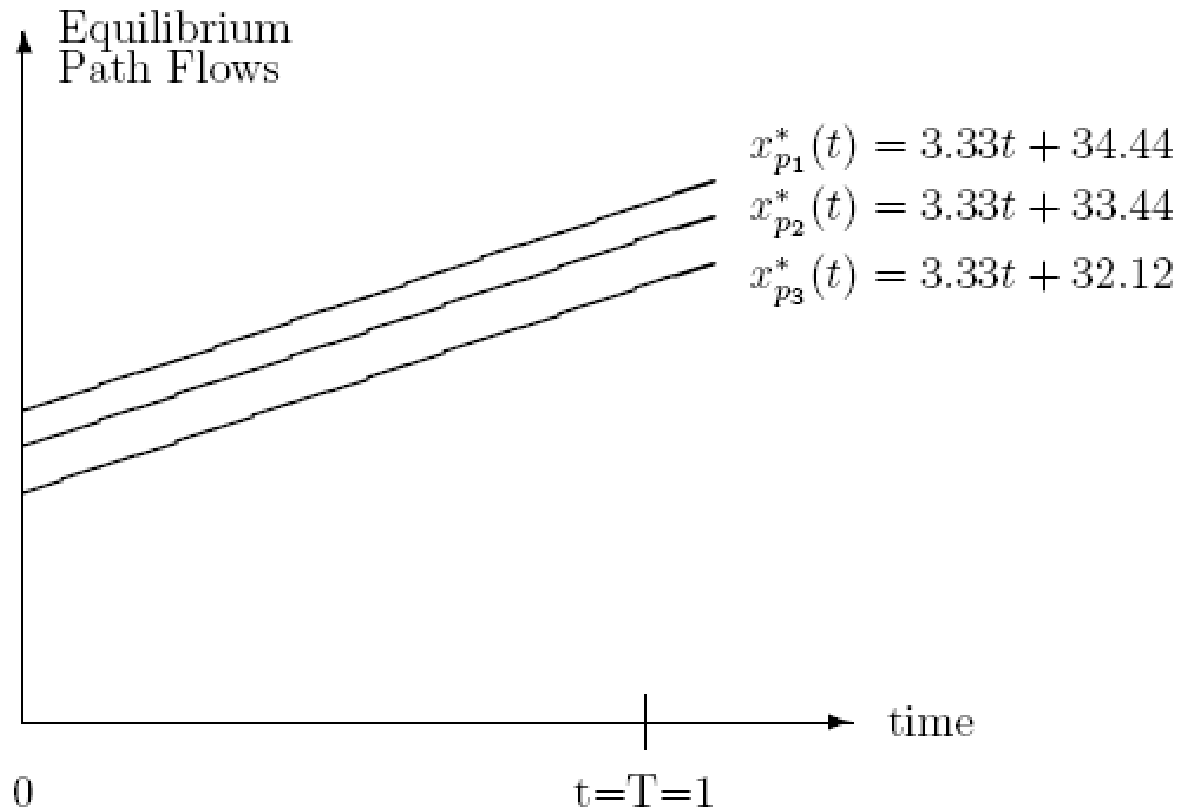
$$x^*_{p_2}(t) = 3.33t + 13.78,$$

$$x^*_{p_3}(t) = 3.33t + 12.45,$$

  - Travel disutility

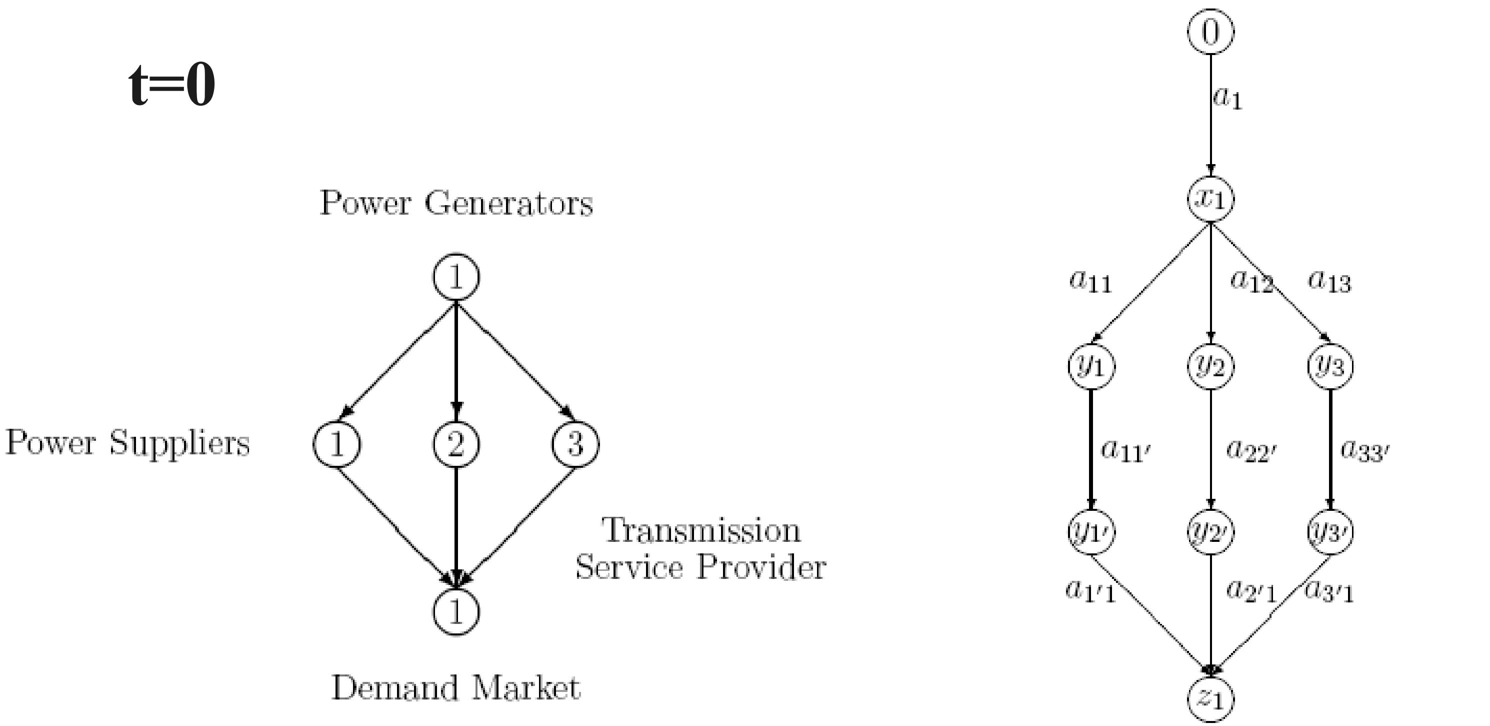
$$\lambda^*_{w_1}(t) = 60t + 255.83, \quad \text{for} \quad t \in [0, T].$$

Time-Dependent Equilibrium Path Flows for Numerical Example 1



The Solution of Numerical Example 1

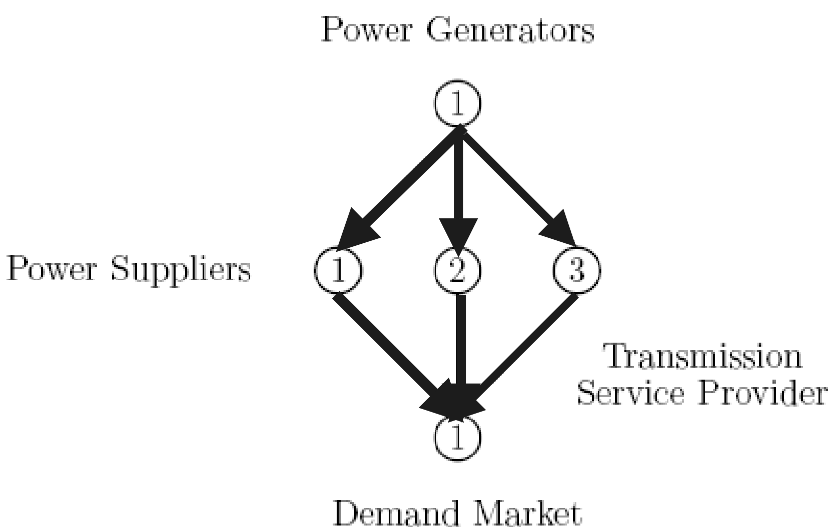
t=0

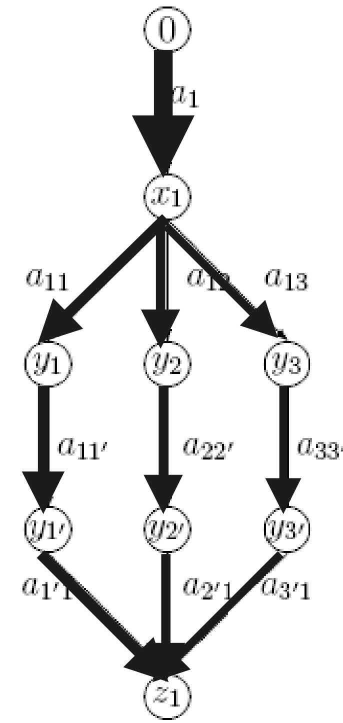


Corresponding Supernetwork

The Solution of Numerical Example 1

t=1/2

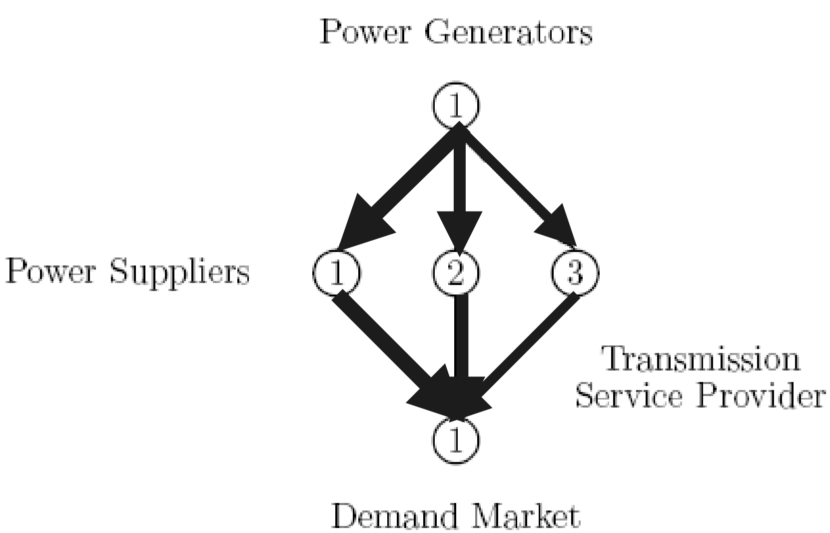


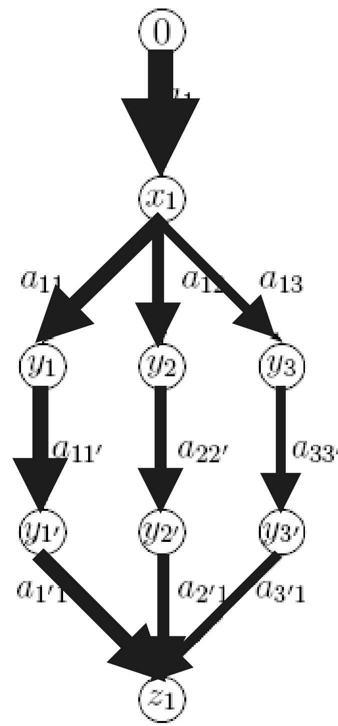


Corresponding Supernetwork

The Solution of Numerical Example 1

t=1





Corresponding Supernetwork

Numerical Example 2

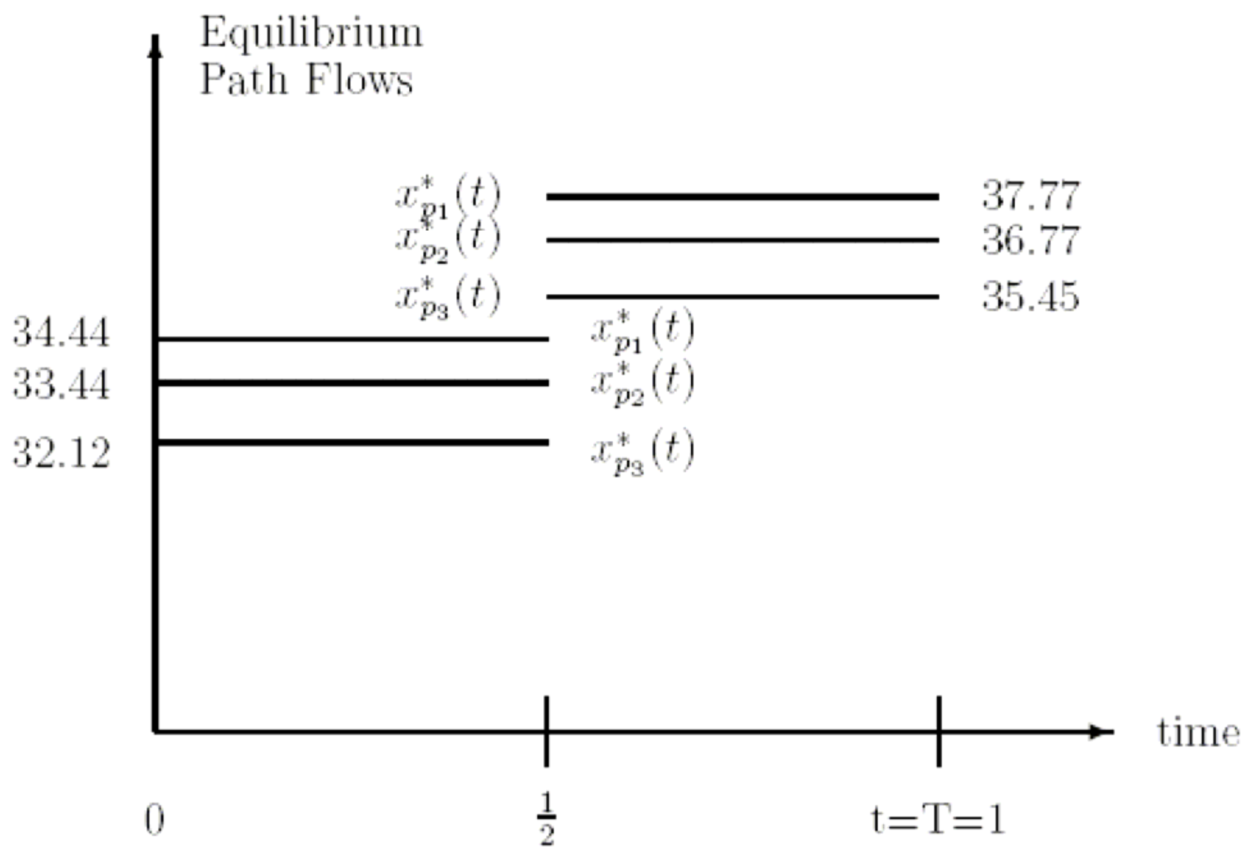
- The network structure and the cost functions are the same as the first example.
- The demand function is the step function:

$$d_1(t) = \begin{cases} 100, & \text{if} \quad 0 < t \leq t_1 = \frac{1}{2}, \\ 110, & \text{if} \quad t_1 < t \leq t_2 = T = 1. \end{cases}$$

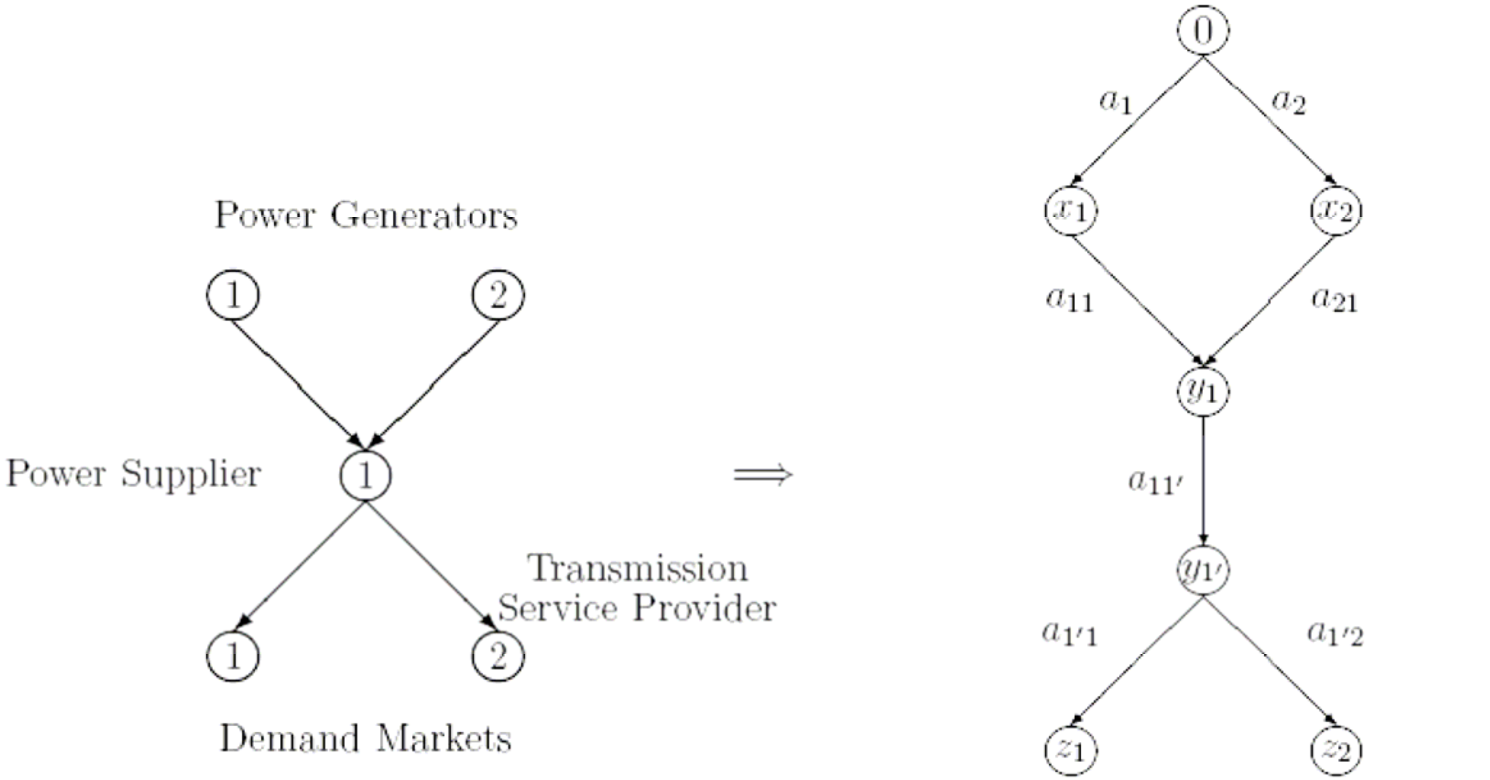
- The explicit solution:

$$x^*(t) = (x^*_{p_1}(t), x^*_{p_2}(t), x^*_{p_3}(t)) = \begin{cases} (34.44, 33.44, 32.12), & \text{if} \quad 0 < t \leq t_1 = \frac{1}{2}, \\ (37.77, 36.77, 35.45), & \text{if} \quad t_1 < t \leq t_2 = 1 = T. \end{cases}$$

Time-Dependent Equilibrium Path Flows for Numerical Example 2



Numerical Example 3



Corresponding Supernetwork

Numerical Example 3

- Generating cost functions

$$f_1(q(t)) = 2.5(q_1(t))^2 + q_1(t)q_2(t) + 2q_1(t), \quad f_2(q(t)) = 2.5(q_2(t))^2 + q_2(t)q_1(t) + 2q_2(t).$$

- Transaction cost functions of the products

$$c_{11}(q_{11}(t)) = .5(q_{11}(t))^2 + 3.5q_{11}(t), \quad c_{21}(q_{21}(t)) = .5(q_{21}(t))^2 + 1.5q_{21}(t).$$

Numerical Example 3

- Operating cost function of the suppliers

$$c_1(Q^1(t)) = .5(q_{11}(t))^2.$$

- Unit transaction costs between the suppliers and the demand markets

$$\hat{c}_{11}^1(Q^2(t)) = q_{11}^1(t) + 1, \quad \hat{c}_{12}^1(Q^2(t)) = q_{12}^1(t) + 1,$$

Numerical Example 3

- Four paths

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}),\ p_2 = (a_2, a_{21}, a_{11'}, a_{1'1}),$$

$$p_3 = (a_1, a_{11}, a_{11'}, a_{1'2}),\ p_4 = (a_2, a_{21}, a_{11'}, a_{1'2})$$

- The time-varying demand functions

$$d_{w_1}(t) = d_1(t) = 100 + 5t, \quad d_{w_2}(t) = d_2(t) = 80 + 4t.$$

The Solution of Numerical Example 3

- Numerical Solution
  - t=0

    $$x^*_{p_1} = 49.90, \quad x^*_{p_2} = 50.10, \quad x^*_{p_3} = 39.90, \quad x^*_{p_4} = 40.10.$$

    $$\lambda^*_{w_1}(t_0) = 915.50 \quad \lambda^*_{w_2}(t_0) = 895.50.$$

  - t=1/2

    $$x^*_{p_1} = 51.15, \quad x^*_{p_2} = 51.35, \quad x^*_{p_3} = 40.90, \quad x^*_{p_4} = 41.10.$$

    $$\lambda^*_{w_1}(t_1) = 938.25, \quad \lambda^*_{w_2}(t_1) = 917.75$$

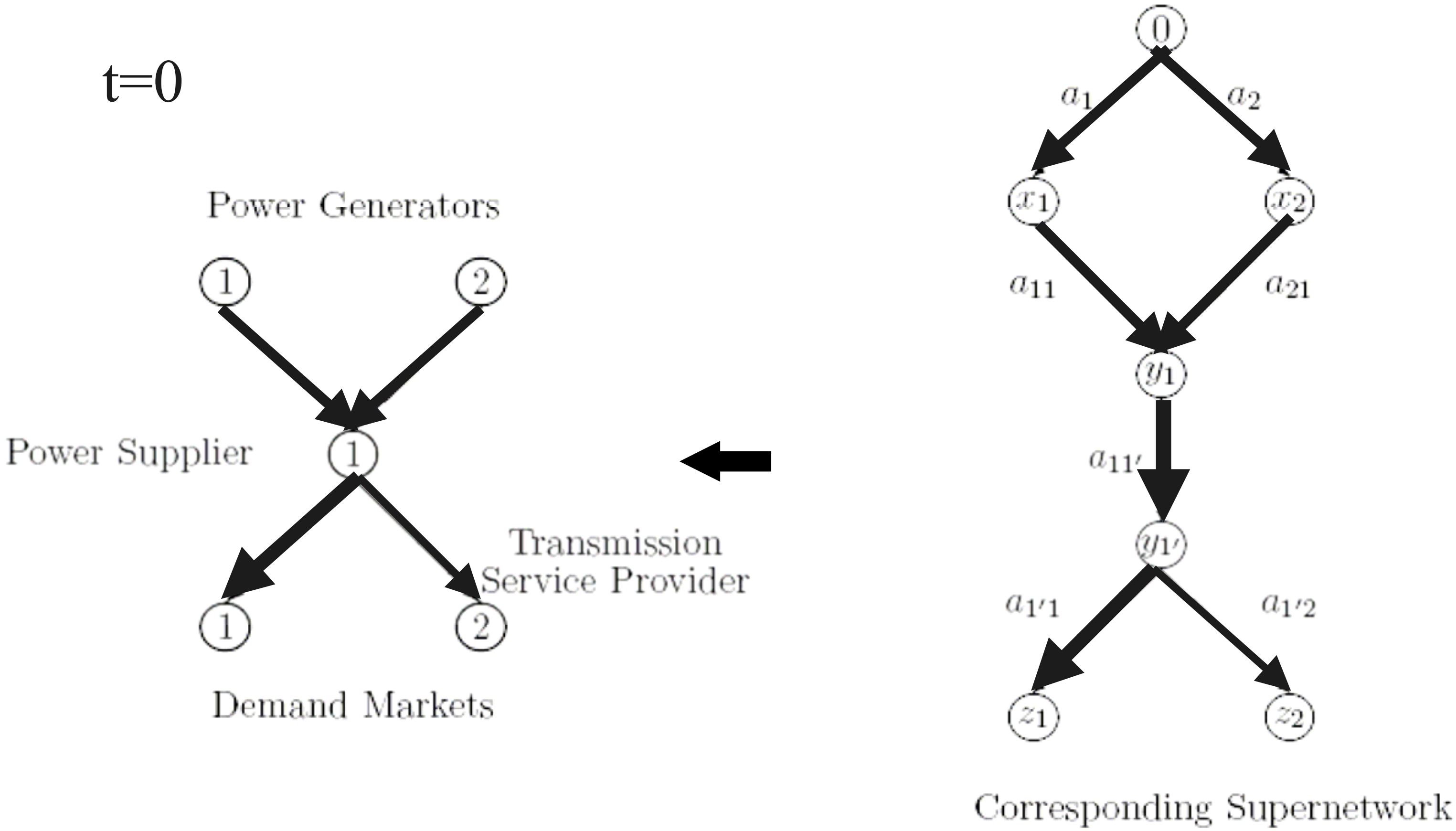
  - t=1

    $$x^*_{p_1} = 52.40, \quad x^*_{p_2} = 52.60, \quad x^*_{p_3} = 41.90, \quad x^*_{p_4} = 42.10.$$

    $$\lambda^*_{w_1}(T) = 961.00, \quad \lambda^*_{w_2}(T) = 940.00.$$

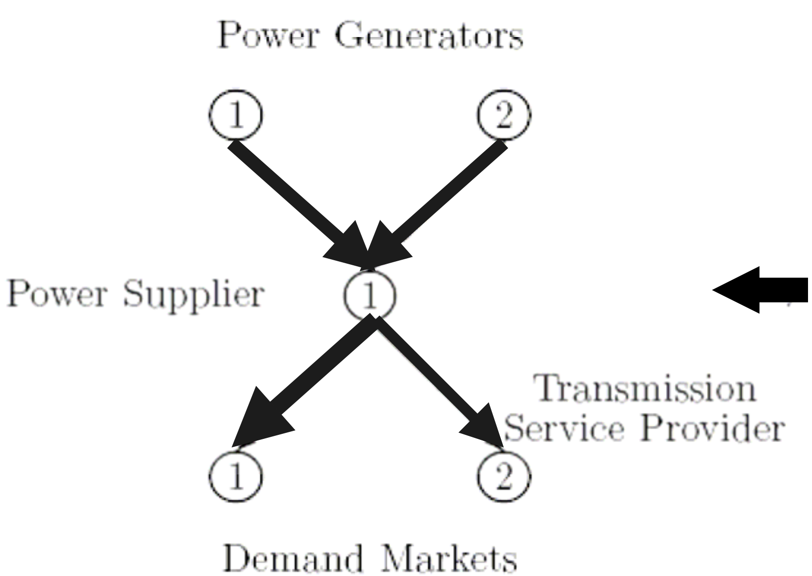
The Solution of Numerical Example 3

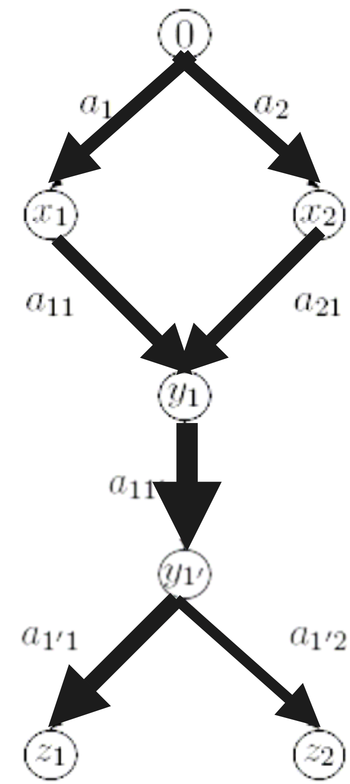
t=0



The Solution of Numerical Example 3

t=1/2

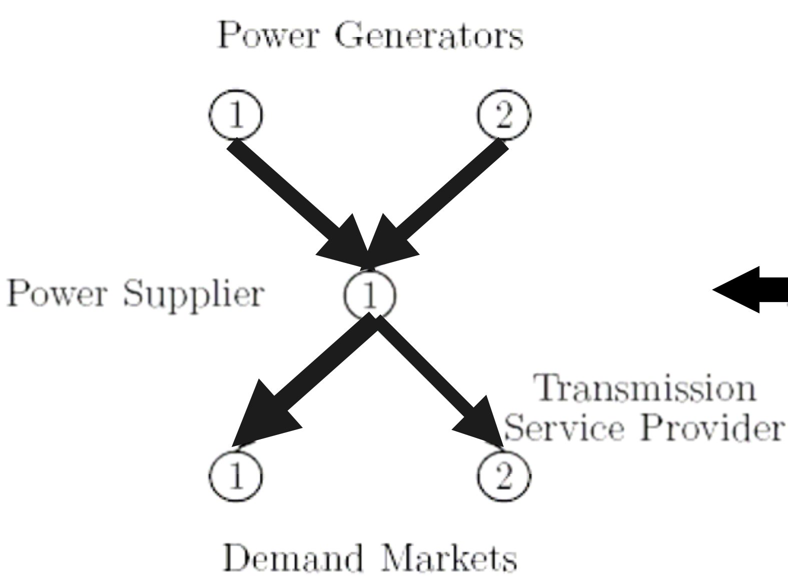


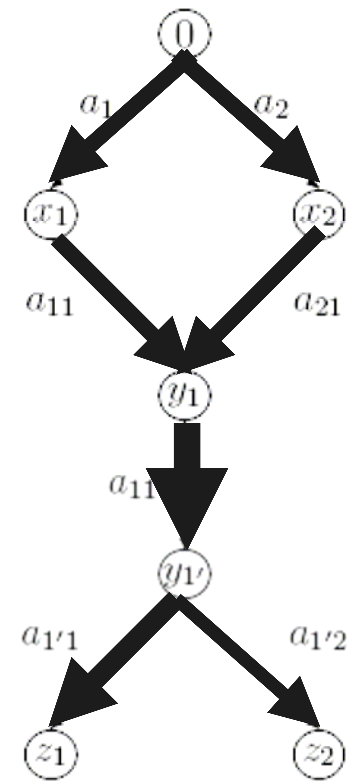


Corresponding Supernetwork

The Solution of Numerical Example 3

t=1





Corresponding Supernetwork

Conclusions

- We established the supernetwork equivalence of the electric power supply chain networks with transportation networks with fixed demands.
- This identification provided a new interpretation of equilibrium in electric power supply chain networks in terms of path flows.
- We utilized this isomorphism in the computation of the electric power supply chain network equilibrium with time-varying demands.

Thank You!

For more information, please see:

The Virtual Center for Supernetworks

http://supernet.som.umass.edu