A Transportation Network Efficiency Measure that Captures Flows, Behavior, and Costs with Applications to Network Component Importance Identification and Vulnerability

Anna Nagurney

Qiang Qiang

Isenberg School of Management

University of Massachusetts – Amherst

13th International Conference on Computing in Economics and Finance

Montréal, Quebec, Canada June 14 - 16, 2007

CEF 2007 MONTRÉAL

Eugene M. Isenberg School of Management

The Virtual Center for Supernetworks

Funding for our research has been provided by:

National Science Foundation

AT&T Foundation

John F. Smith Memorial Fund - University of Massachusetts at Amherst

THE ROCKEFELLER FOUNDATION

RADCLIFFE INSTITUTE FOR ADVANCED STUDY
HARVARD UNIVERSITY

Network Vulnerability

- Recent disasters have demonstrated the importance as well as the vulnerability of network systems.
- For example:
  - Hurricane Katrina, August 23, 2005
  - The biggest blackout in North America, August 14, 2003
  - 9/11 Terrorist Attacks, September 11, 2001

Earthquake Damage

prcs.org.pk

Tsunami

letthesunshinein.wordpress.com

Storm Damage

www.srh.noaa.gov

Infrastructure Collapse

www.10-7.com

An Urgent Need for a Network Efficiency/Performance Measure

In order to be able to assess the performance/efficiency of a network, it is imperative that appropriate measures be devised.

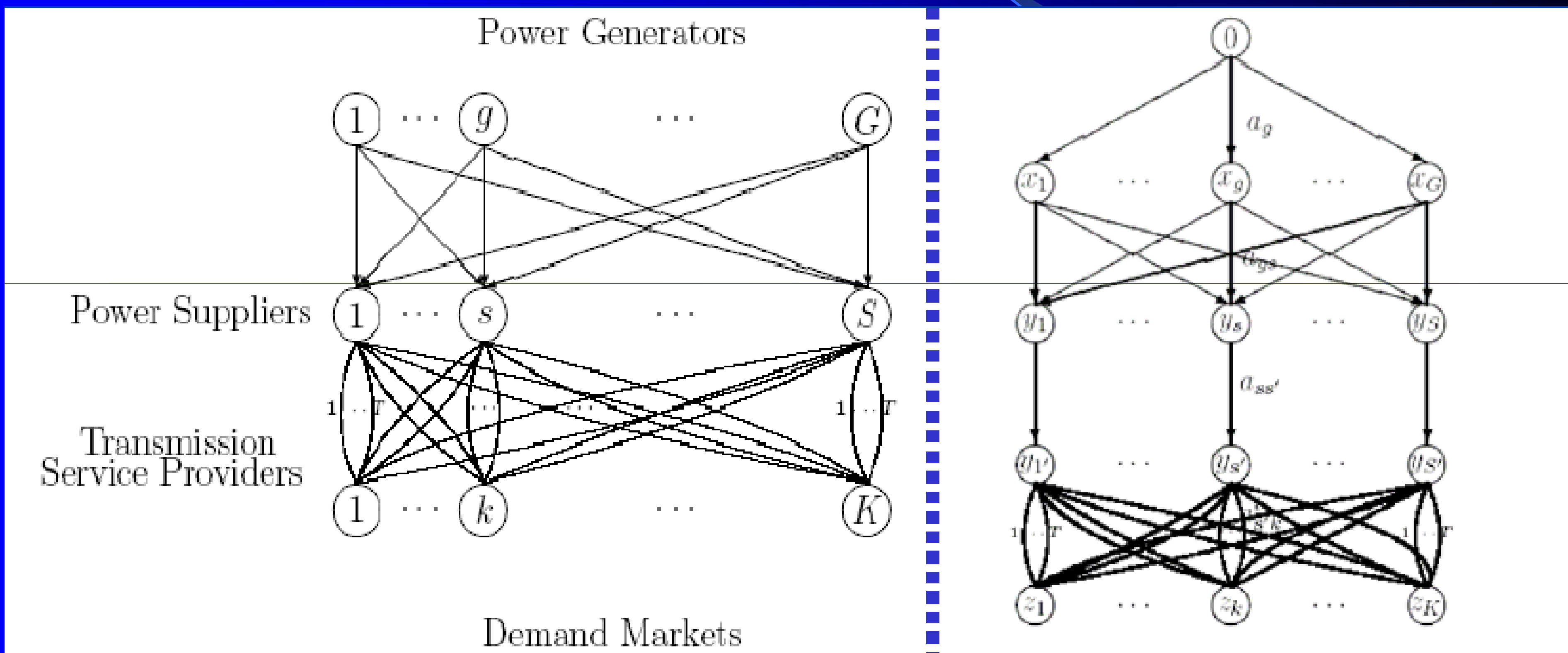
Appropriate network measures can assist in the identification of the importance of network components, that is, nodes and links, and the associated rankings. Such rankings can be very helpful in the case of the determination of network vulnerabilities as well as when to reinforce/enhance security.

Transportation Network Equilibrium Paradigm

It has been recently shown that, as hypothesized over 50 years ago by Beckmann, McGuire, and Winsten (1956), that electric power generation and distribution networks can be reformulated and solved as transportation networks, Wu, Nagurney, Liu, and Stranlund, *Transportation Research D* (2006), Nagurney et al., *Transportation Research D*, in press.

It has been demonstrated that financial networks with intermediation can be reformulated and solved as transportation network problems; Liu and Nagurney, *Computational Management Science*, in press.

The Transportation Network Equilibrium Reformulation of Electric Power Supply Chain Networks

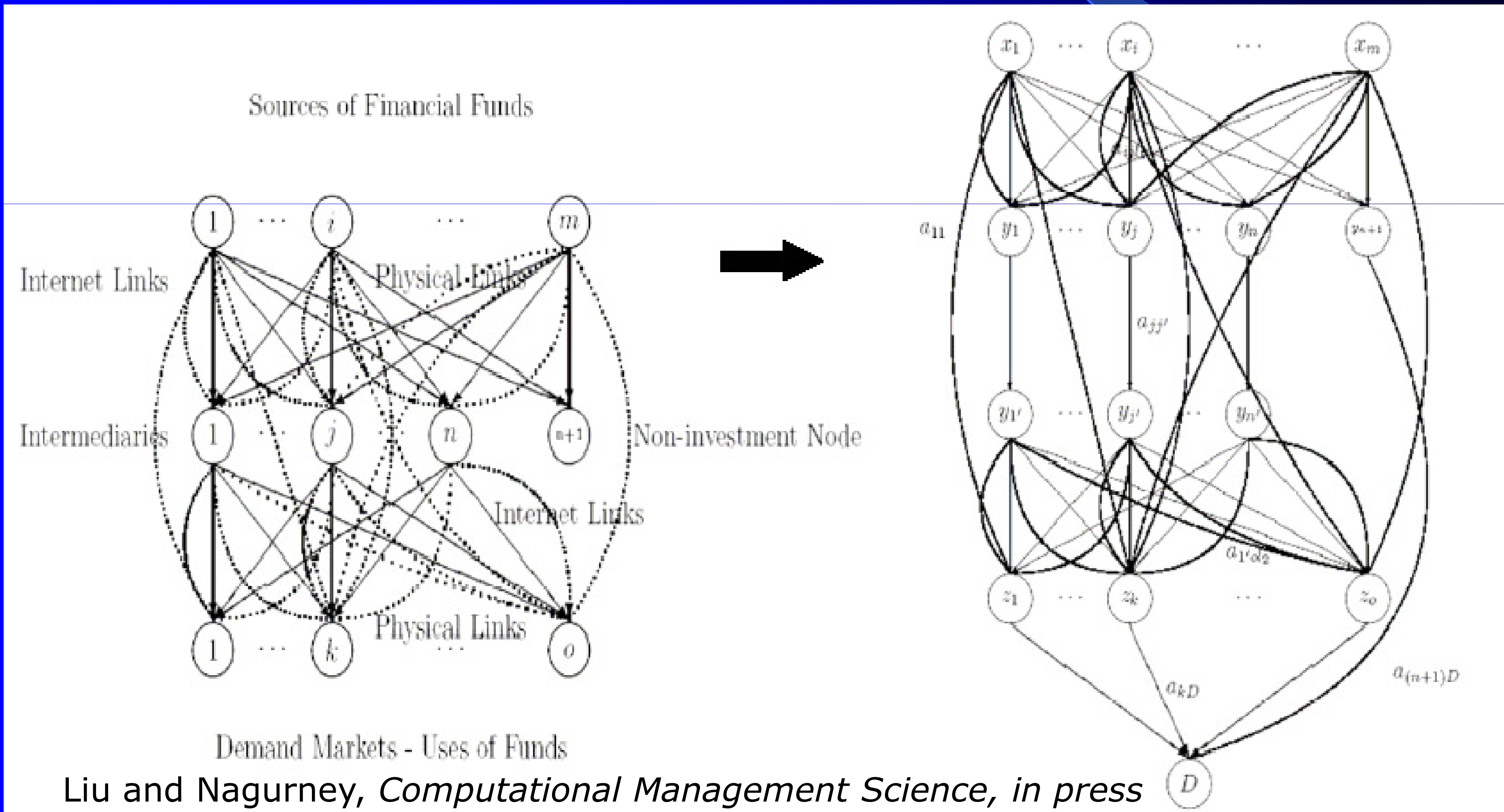


Electric Power Supply Chain Network

Transportation Network

Nagurney et al, to appear in *Transportation Research E*

The Transportation Network Equilibrium Reformulation of the Financial Network Equilibrium Model with Intermediation



Liu and Nagurney, *Computational Management Science, in press*

Transportation Network Equilibrium Problem

Consider a general network $G = [N, L]$, where N denotes the set of nodes, and L the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote a path consisting of a sequence of links connecting an O/D pair. P_w denotes the set of paths, assumed to be acyclic, connecting the O/D pair of nodes w and P the set of all paths.

Let x_p represent the flow on path p and f_a the flow on link a. The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap},$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. This expression states that the load on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Moreover, if we let d_w denote the demand associated with O/D pair w, then we must have that

$$d_w = \sum_{p \in P_w} x_p,$$

where $x_p \geq 0$, $\forall p$, that is, the sum of all the path flows between an origin/destination pair w must be equal to the given demand d_w.

Let c_a denote the user cost associated with traversing link a, which is assumed to be continuous, and C_p the user cost associated with traversing the path p. Then

$$C_p = \sum_{a \in L} c_a \delta_{ap}.$$

In other words, the cost of a path is equal to the sum of the costs on the links comprising the path.

Transportation science has historically been the discipline that has pushed the frontiers in terms of methodological developments for such problems (which are often large-scale) beginning with the work of Beckmann, McGuire, and Winsten (1956).

Definition: Transportation Network Equilibrium

A route flow pattern $x^* \in K$ is said to be a transportation network equilibrium (according to Wardrop's (1952) first principle) if only the minimum cost routes are used (that is, have positive flow) for each O/D pair. The state can be expressed by the following equilibrium conditions which must hold for every O/D pair $w \in W$, every path $p \in P_w$:

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0. \end{cases}$$

As shown by Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969), if the user link cost functions satisfy the symmetry property that $[\frac{\partial c_b}{\partial c_a} = \frac{\partial c_a}{\partial c_b}]$ for all links a, b in the network then the solution to the above network equilibrium problem can be reformulated as the solution to an associated optimization problem. For example, if we have that $c_a = c_a(f_a)$, $\forall a \in L$, then the solution can be obtained by solving:

$$\text{Minimize} \quad \sum_{a \in L} \int_0^{f_a} c_a(y)dy$$

subject to:

$$d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W,$$

$$f_a = \sum_{p \in P} x_p, \quad \forall a \in L,$$

$$x_p \geq 0, \quad \forall p \in P.$$

If no such symmetry assumption holds for the user link costs functions, then the equilibrium conditions can no longer be reformulated as an associated optimization problem and the equilibrium conditions are formulated and solved as a *variational inequality problem!*

Smith (1979), Dafermos (1980)

VI Formulation of Transportation Network Equilibrium (Dafermos (1980), Smith (1979))

A traffic path flow pattern satisfies the above equilibrium conditions if and only if it satisfies the variational inequlity problem: determine $x^* \in K$, such that

$$\sum_p C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in K.$$

Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above.

In particular, the finite-dimensional variational inequality problem is to determine $x^* \in K \subset R^n$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

where $\langle \cdot, \cdot \rangle$ denoted the inner product in R^n and K is closed and convex.

Recent Literature on Network Vulnerability

- Latora and Marchiori (2001, 2002, 2004)
- Barrat, Barthélemy and Vespignani (2005)
- Dall'Asta, Barrat, Barthélemy and Vespignani (2006)
- Chassin and Posse (2005)
- Holme, Kim, Yoon and Han (2002)
- Sheffi (2005)
- Taylor and D'este (2004)
- Jenelius, Petersen and Mattson (2006)
- Murray-Tuite and Mahmassani (2004)

The Network Efficiency Measure of Latora and Marchiori (2001)

Latora and Marchiori (2001) proposed a network efficiency measure (the L-M measure) as follows:

Definition : The L-M Measure

The network performance/efficiency measure, $E(G)$, according to Latora and Marchiori (2001) for a given network topology G, is defined as:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$$

where n is the number of nodes in the network and d_{ij} is the shortest path length between node i and node j.

The Nagurney and Qiang Network Efficiency Measure

Nagurney and Qiang (2007a) (the N-Q Measure) proposed a network efficiency measure for networks with fixed demand, which captures demand and flow information under the network equilibrium.

Definition : The N-Q Measure

The network performance/efficiency measure, $\mathcal{E}(G,d)$, according to Nagurney and Qiang (2007), for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W},$$

where recall that n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w

The L-M Measure vs. the N-Q Measure

Theorem

If positive demands exist for all pairs of nodes in the network G, and each of these demands is equal to 1 and if d_{ij} is set equal to λ_w, where $w = (i, j)$, for all $w \in W$ then the proposed network efficiency measure and the L-M measure are one and the same.

Importance of a Network Component

Definition : Importance of a Network Component According to the L-M Measure

The importance of a network component $g \in G$, $\bar{I}(g)$, is measured by the network efficiency drop, determined by the L-M measure, after g is removed from the network:

$$\bar{I}(g) = \frac{\triangle E}{E(G)} = \frac{E(G) - E(G-g)}{E(G)}.$$

where $G - g$ is the resulting network after component g is removed from network G.

Definition : Importance of a Network Component According to the N-Q Measure

The importance of a network component $g \in G$, $I(g)$, is measured by the relative network efficiency drop, determined by the N-Q measure, after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G,d) - \mathcal{E}(G-g,d)}{\mathcal{E}(G,d)},$$

where $G - g$ is the resulting network after component g is removed from network G.

The Approach to Study the Importance of Network Components

The elimination of a link is represented in the N-Q measure by the removal of that link while the removal of a node is managed by removing the links entering and exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity.

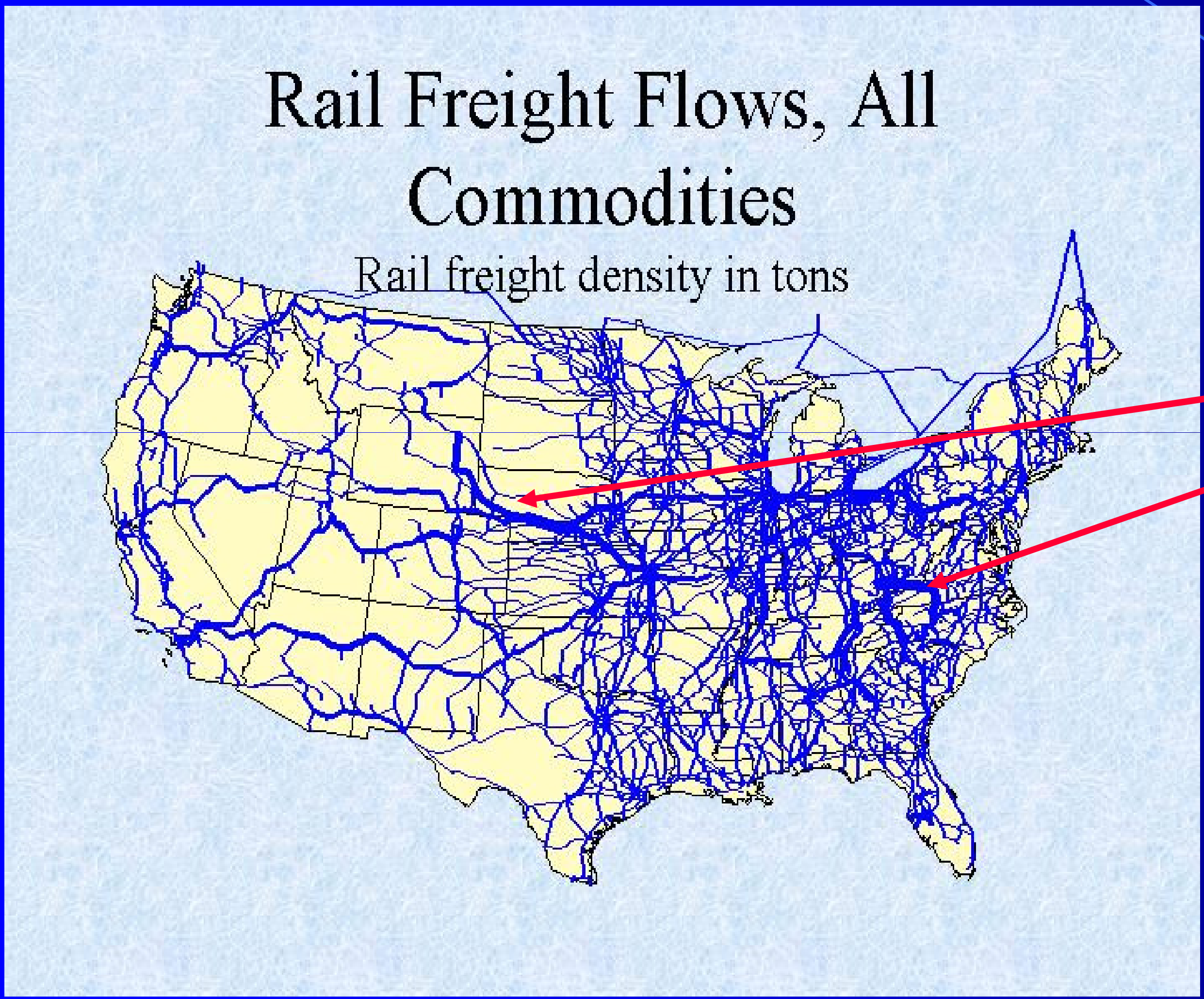
Hence, our measure is well-defined even in the case of disconnected networks.

Major Advantages of the N-Q Measure over the L-M Measure

- The N-Q measure generalizes the L-M measure by capturing the flows, demand and user behavior information of the network besides the network topology structure.

- It has been shown that real-life networks displayed distinct disparities between topological properties and the flow patterns.

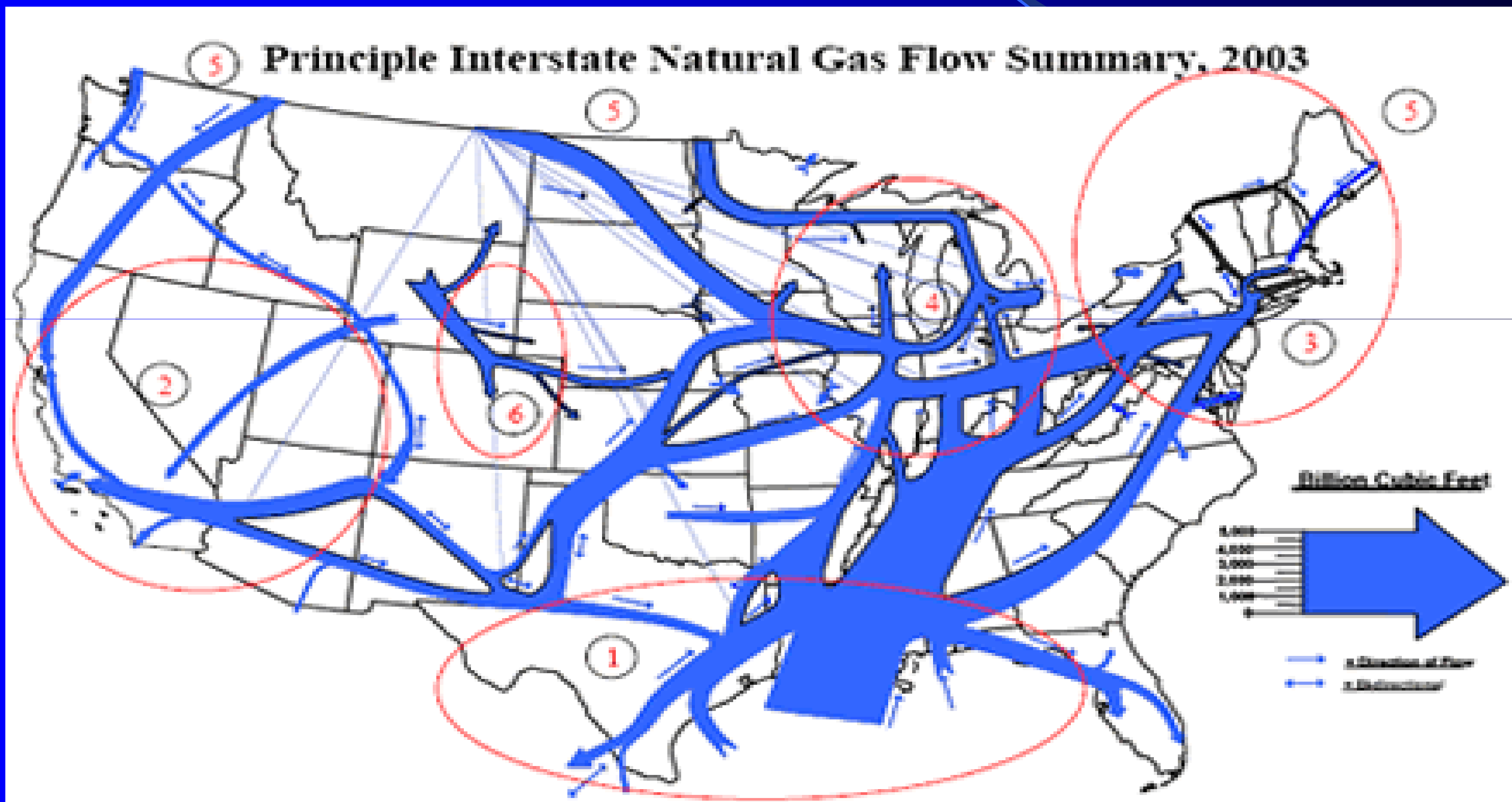
Railway Network in U.S.A



Some links are heavily used.

U.S. Department of Transportation
ops.fhwa.dot.gov

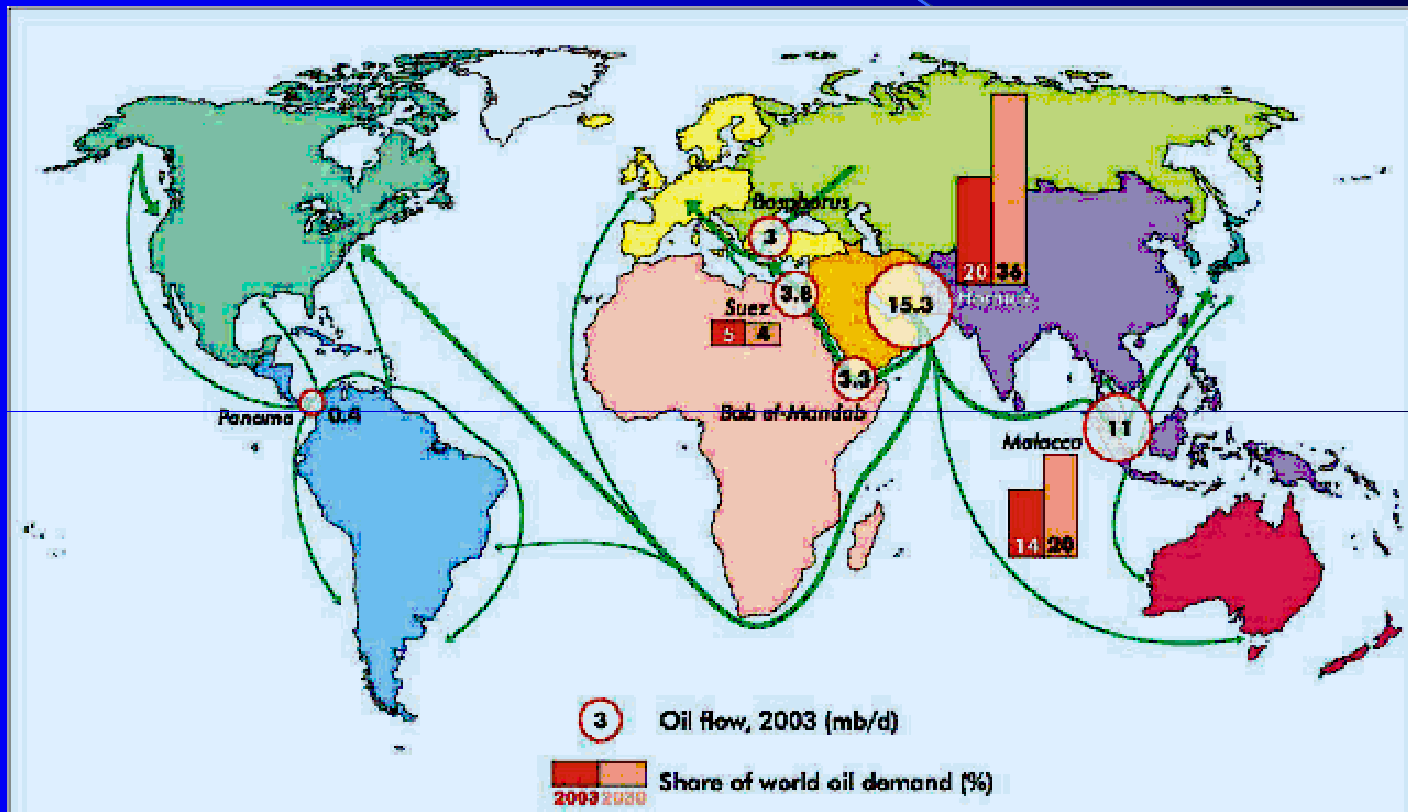
Natural Gas Pipeline Network in USA



Energy Information Administration, U.S. Department of Energy

www.eia.doe.gov

World Oil Trading Network



Treasurer of the Commonwealth Australia

www.treasurer.gov.au

Link Importance Indicators

Definition : Link Importance Indicators According to Jenelius, Petersen, and Mattsson (2006)

In a network G, the global importance, I^1, the demand-weighted importance, I^2, and the relative unsatisfied demand, I^3, of link $k \in G$ are defined, respectively, as follows:

$$I^1(k) = \frac{1}{n_W} \sum_{w \in W} (\lambda_w(G-k) - \lambda_w(G)),$$

$$I^2(k) = \frac{\sum_{w \in W} d_w(\lambda_w(G-k) - \lambda_w(G))}{\sum_{w \in W} d_w},$$

$$I^3(k) = \frac{\sum_{w \in W} u_w(G-k)}{\sum_{w \in W} d_w},$$

where $\lambda_w(G)$ is the original equilibrium cost of O/D pair w while $\lambda_w(G-k)$ is the equilibrium cost of O/D pair w after link k is removed; $u_w(G-k)$ is the unsatisfied demand for O/D pair w after link k is removed.

Advantages of the N-Q Measure over Link Importance Indicators

- The N-Q measure is unified and can be applied to any network component, be it a node, or a link of a set of nodes and links;
- The N-Q measure is independent of whether the network is disconnected or not.

Our Research on Network Efficiency and Network Vulnerability

A Network Efficiency Measure with Application to Critical Infrastructure Networks, Nagurney and Qiang (2007a), to appear in *Journal of Global Optimization.*

A Transportation Network Efficiency Measure that Captures Flows, Behavior, and Costs with Applications to Network Component Importance Identification and Vulnerability, Nagurney and Qiang (2007b), *Proceedings of the POMS 18th Annual Conference*, May 4 to May 7, 2007.

A Unified Network Performance Measure with Importance Identification and the Ranking of Network Components (2007), Qiang and Nagurney, *Optimization Letters,* in press.

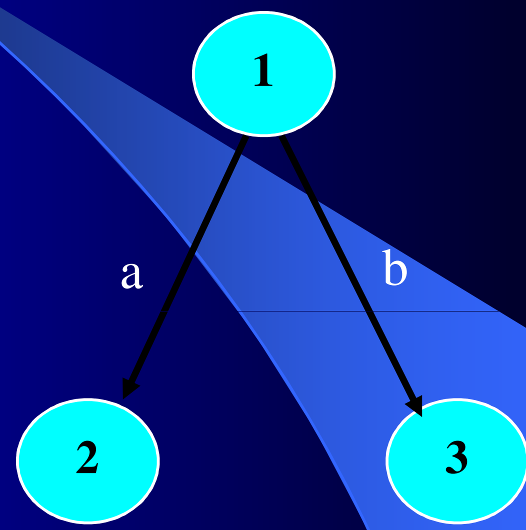
Example 1

Assume a network with two O/D pairs: $\boldsymbol{w_1}$=(1,2) and $\boldsymbol{w_2}$=(1,3) with demands given, respectively, by $\boldsymbol{d_{w1}}$=100 and $\boldsymbol{d_{w2}}$=20. The path for each O/D pair is: for $\boldsymbol{w_1}$, $\boldsymbol{p_1 = a}$; for $\boldsymbol{w_2}$, $\boldsymbol{p_2 = b}$.

The equilibrium path flows are $x_{p_1}^* =$ 100, $x_{p_2}^*$=20.

The equilibrium path travel cost is

$C_{p_1} = C_{p_2}$=20.



$$c_a(f_a)=0.01 f_a+19$$
$$c_b(f_b)=0.05 f_b+19$$

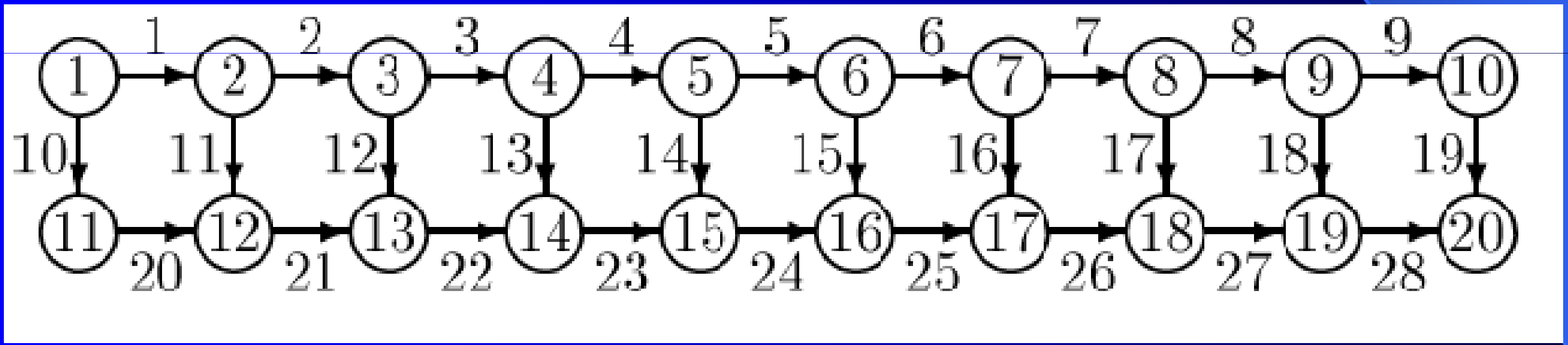
Importance and Ranking of Links and Nodes

Link	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure	Importance Value from I^3	Importance Ranking from I^3
a	0.8333	1	0.5000	1	0.8333	1
b	0.1667	2	0.5000	1	0.1667	2

Node	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure
1	1.0000	1	1.0000	1
2	0.8333	2	0.5000	2
3	0.1667	3	0.5000	2

Example 2

The network topology is the following:



$w_1 = (1,19)$, $w_2 = (1,20)$

$d_{w_1} = d_{w_2} = 100$

Link Cost Functions

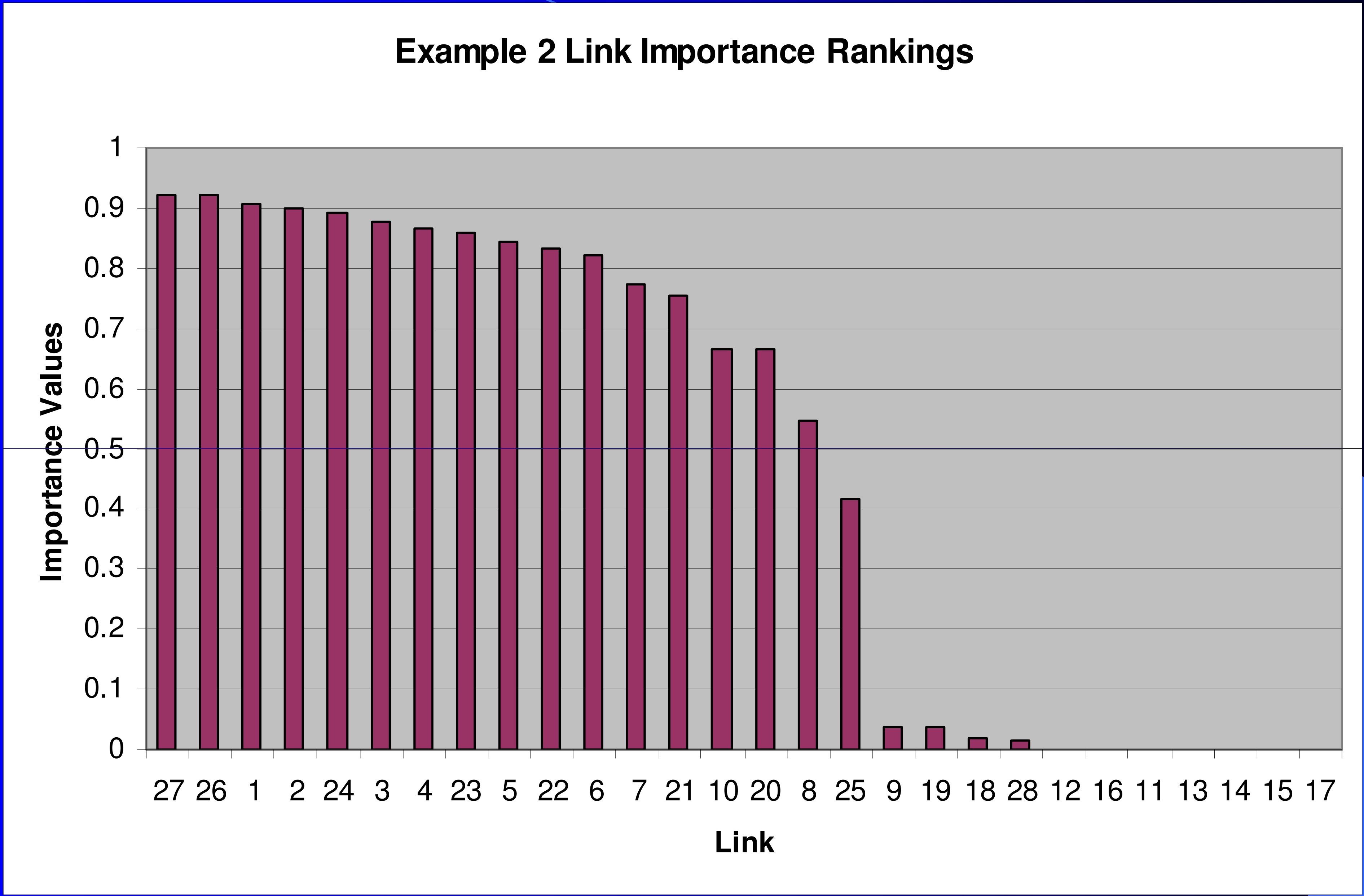
Link a	Link Cost Function $c_a(f_a)$
1	$.00005 f_1^4 + 5 f_1 + 500$
2	$.00003 f_2^4 + 4 f_2 + 200$
3	$.00005 f_3^4 + 3 f_3 + 350$
4	$.00003 f_4^4 + 6 f_4 + 400$
5	$.00006 f_5^4 + 6 f_5 + 600$
6	$7 f_6 + 500$
7	$.00008 f_7^4 + 8 f_7 + 400$
8	$.00004 f_8^4 + 5 f_8 + 650$
9	$.00001 f_9^4 + 6 f_9 + 700$
10	$4 f_{10} + 800$
11	$.00007 f_{11}^4 + 7 f_{11} + 650$
12	$8 f_{12} + 700$
13	$.00001 f_{13}^4 + 7 f_{13} + 600$
14	$8 f_{14} + 500$

Link a	Link Cost Function $c_a(f_a)$
15	$.00003 f_{15}^4 + 9 f_{15} + 200$
16	$8 f_{16} + 300$
17	$.00003 f_{17}^4 + 7 f_{17} + 450$
18	$5 f_{18} + 300$
19	$8 f_{19} + 600$
20	$.00003 f_{20}^4 + 6 f_{20} + 300$
21	$.00004 f_{21}^4 + 4 f_{21} + 400$
22	$.00002 f_{22}^4 + 6 f_{22} + 500$
23	$.00003 f_{23}^4 + 9 f_{23} + 350$
24	$.00002 f_{24}^4 + 8 f_{24} + 400$
25	$.00003 f_{25}^4 + 9 f_{25} + 450$
26	$.00006 f_{26}^4 + 7 f_{26} + 300$
27	$.00003 f_{27}^4 + 8 f_{27} + 500$
28	$.00003 f_{28}^4 + 7 f_{28} + 650$

Importance and Ranking of Links

Link a	Importance Value	Importance Ranking
1	0.9086	3
2	0.8984	4
3	0.8791	6
4	0.8672	7
5	0.8430	9
6	0.8226	11
7	0.7750	12
8	0.5483	15
9	0.0362	17
10	0.6641	14
11	0.0000	22
12	0.0006	20
13	0.0000	22
14	0.0000	22

Link a	Importance Value	Importance Ranking
15	0.0000	22
16	0.0001	21
17	0.0000	22
18	0.0175	18
19	0.0362	17
20	0.6641	14
21	0.7537	13
22	0.8333	10
23	0.8598	8
24	0.8939	5
25	0.4162	16
26	0.9203	2
27	0.9213	1
28	0.0155	19



Example 2 Link Importance Rankings
Importance Values
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
27 26 1 2 24 3 4 23 5 22 6 7 21 10 20 8 25 9 19 18 28 12 16 11 13 14 15 17
Link

The Advantages of the Nagurney and Qiang Network Efficiency Measure

- It captures flows, costs, and behavior of travelers, in addition to network topology;
- The resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;
- It can be used to identify the importance (and ranking) of either nodes, or links, or both;
- It can be applied to assess the efficiency/performance of a wide range of critical infrastructure networks;
- It is the unified measure that can be used to assess the network efficiency with either fixed or elastic demands.

The Virtual Center for Supernetworks

Supernetworks for Optimal Decision-Making and Improving the Global Quality of Life

Home | About | Background | Activities | Publications | Media | Links | What's New | Search

The Virtual Center for Supernetworks at the Isenberg School of Management, under the directorship of Anna Nagurney, the John F. Smith Memorial Professor, is an interdisciplinary center, and includes the Supernetworks Laboratory for Computation and Visualization.

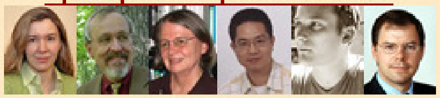
Mission: The mission of the Virtual Center for Supernetworks is to foster the study and application of supernetworks and to serve as a resource to academia, industry, and government on networks ranging from transportation, supply chains, telecommunication, and electric power networks to economic, environmental, financial, knowledge and social networks.

The Applications of Supernetworks Include: multimodal transportation networks, critical infrastructure, energy and the environment, the Internet and electronic commerce, global supply chain management, international financial networks, web-based advertising, complex networks and decision-making, integrated social and economic networks, network games, and network metrics.

Announcements and Notes from the Center Director Professor Anna Nagurney

Updated: April 12, 2007

UMass Amherst INFORMS Student Chapter Spring 2007 Speaker Series



Press Release

NEW!

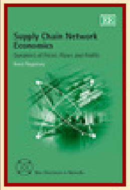
See New Papers on: Network Vulnerability and Disruptions, Dynamic Internet Traffic and the Braess paradox, and Electric Power Supply Chain Networks!

Radcliffe Exploratory Seminar on Dynamic Networks: Behavior, Optimization and Design

Presentations and Papers

RADCLIFFE INSTITUTE FOR ADVANCED STUDY
HARVARD UNIVERSITY

Photos from the Seminar



Supply Chain Network Economics

Press Release

You are visitor number

39,298

to the Virtual Center for Supernetworks.

Google

Google Search

Thank You!

For more information, see

http://supernet.som.umass.edu

Eugene M. Isenberg School of Management

The Virtual Center for Supernetworks