Lecture 5
System-Optimizing Transportation Network

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System-Optimizing Transportation Network

Central authority can route traffic according to his will **(users can't make their own choices).**

Criterion or Objective

To minimize total cost in the network.

The total cost on a link =

$\hat{c}_a(f_a) = c_a(f_a) \times f_a$

$c_a(f_a)$: user cost $\qquad$ f_a : total number of users on a.

Hence the total cost on a network can be expressed as:

$$\sum_a \hat{c}_a(f_a)$$

The system optimization (S-O) problem can, hence, be expressed as:

$$\text{Minimize} \sum_a \hat{c}_a(f_a) \qquad [\text{ the total cost}]$$

subject to the constraints:

$$d_w = \sum_{p \in P_w} F_p, \text{ for all O/D pairs } w.$$

$$f_a = \sum_p F_p \delta_{ap}, \text{ for all links } a.$$

$$F_p \geq 0, \text{ for all } p.$$

Note: The constraints are identical to those in the U-O problem.

S-O conditions

Theorem

A link load pattern f induced by a path flow pattern F is system optimizing if and only if there exists an ordering of paths:

$p_1, ..., p_s, p_{s+1}, ..., p_m$

that connect O/D pair w, such that

$$\hat{C}'_{p_1}(f) = ... = \hat{C}'_{p_{s'}}(f) = \lambda_w \leq \hat{C}'_{p_{s'+1}}(f) \leq ... \leq \hat{C}'_{p_{m'}}(f)$$

$$F_{p_r} > 0; r = 1, ..., s'$$
$$F_{p_r} = 0; r = s' + 1, ..., m',$$

where

$$\hat{C}'_{p_1}(f) = \sum_a \hat{c}'_a(f_a)\delta_{ap} = \sum_a \frac{\partial \hat{c}_a(f_a)}{\partial f_a}\delta_{ap}$$

$\hat{C}'_{p_1}(f)$: marginal of total cost on a path

$\hat{c}'_a(f_a)$: marginal of total cost on a link

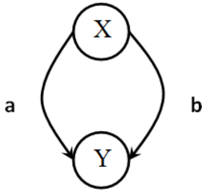
The above condition must be satisfied for every O/D pair in the network.

These conditions are equivalent to the Kuhn-Tucker conditions of the optimization problem.

If the marginal cost functions are **strictly** increasing functions of the flows on the links, we are guaranteed a **unique** S-O link flow pattern.

Example

Find the U-O and the S-O pattern for the following network:



$d_{xy} = 100$

user link cost functions:

$c_a(f_a) = 3f_a + 1000$
$c_b(f_b) = 2f_b + 1500$

Recall that a flow pattern would be U-O if:

$c_a = c_b$ and $f_a > 0,\ f_b > 0,$
or $c_a \geq c_b$ and $f_b > 0,\ f_a = 0,$
or $c_a \leq c_b$ and $f_a > 0,\ f_b = 0.$

Let's use the algorithm:

- Sort the h_a's : $1000 < 1500$
- Compute

$$v_w^1 = \frac{d_w + \frac{h_{a_1}}{g_{a_1}}}{\frac{1}{g_{a_1}}} = \frac{100 + \frac{1000}{3}}{\frac{1}{3}} = 1300.$$

- Check:

Is: $1000 < 1300 \leq 1500$; *Yes*!

Critical $s = 1$, so

$$f_a = F_{p_1} = \frac{v_w^1 - h_{a_1}}{g_{a_1}} - \frac{1300-1000}{3} = 100 \; f_b = F_{p_2} = 0$$

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A flow pattern would be S-O if:

$$\hat{c}'_a = \hat{c}'_b \text{ and } f_a > 0, f_b > 0,$$
$$\text{or } \hat{c}'_a \geq \hat{c}'_b \text{ and } f_b > 0, f_a = 0,$$
$$\text{or } \hat{c}'_a \leq \hat{c}'_b \text{ and } f_a > 0, f_b = 0.$$

For this example, let's see if both paths (which are links here) can be used.

Then we must have: $\hat{c}'_a = \hat{c}'_b$

How do we form these?

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$$\hat{c}_a(f_a) = (3f_a + 1000) \times f_a = 3f_a^2 + 1000f_a$$
$$\hat{c}_b(f_b) = (2f_b + 1500) \times f_b = 2f_b^2 + 1500f_b$$

Then

$$\hat{c}'_a = \frac{\partial \hat{c}_a}{\partial f_a} = 6f_a + 1000$$
$$\hat{c}'_b = \frac{\partial \hat{c}_b}{\partial f_b} = 4f_b + 1500$$

What else do we know?

$$f_a + f_b = d_w = 100 \Rightarrow f_b = 100 - f_a$$

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Hence,

$\hat{c}'_a(f_a) = \hat{c}'_b(f_b)$ means that:

$6f_a + 1000 = 4(100 - f_a) + 1500$

$6\text{f}_a + 1000 = 1800 - 4f_a$

$10\text{f}_a = 900, \;\; f_a = 90; \;\; f_b = 10.$

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$*$ The S-O pattern is distinct from the U-O pattern.

The total cost under the **S-O pattern** is
$c_a = 1270; \; c_b = 1520.$

Total cost $= \hat{c}_a + \hat{c}_b = c_a \times f_a + c_b \times f_b =$
$= (1270)90 + (1520)10 = 129,500.$

The total cost under the **U-O pattern** is:
$c_a = 1300; \; c_b = 1500$

Total cost $= \hat{c}_a + \hat{c}_b = c_a \times f_a + c_b \times f_b =$
$= 1300(100) + 1500(0) = 130,000.$

Note : Total Cost under the S-O pattern is less than the Total Cost under the U-O pattern.

$$129{,}500 < 130{,}000 \; !$$

The Exact Equilibration Algorithm (S-O)

- Sort the h_{a_i}'s in non-descending order and relabel accordingly. Set $r = 1$ and $h_{a_{m+1}} = \infty$.
- Compute

$$\lambda_w^r = \frac{d_w + \sum_{i=1}^{r} \frac{h_{a_i}}{2g_{a_i}}}{\sum_{i=1}^{r} \frac{1}{2g_{a_i}}}$$

The Exact Equilibration Algorithm (S-O) (continued)

- Check

  If

$$h_{a_r} < \lambda_w^r \leq h_{a_{r+1}} \quad ,$$

then STOP.

Set the critical $s = r$;

$$F_{p_r} = \frac{\lambda_w^r - h_{a_r}}{2g_{a_r}}; \quad r = 1, ..., s$$

$$F_{p_r} = 0; \quad r = s + 1, ..., m.$$

Else set $r = r + 1$ and goto Step 2.

U-O Condition

For each O/D pair w , there exists an ordering:

$$C_{p_1}(f) = ... = C_{p_s}(f) = v_w \leq C_{p_{s+1}}(f) \leq ... \leq C_{p_m}(f)$$

$F_{p_r} > 0; \quad r = 1, ..., s.$

$F_{p_r} = 0; \quad r = s + 1, ..., m.$

Here user costs on used paths are "equilibrated or equal".

S-O Conditions

For each O/D pair w , there exists an ordering:

$$\hat{C}'_{p_1}(f) = ... = \hat{C}'_{p_{s'}}(f) = \lambda_w \leq \hat{C}'_{p_{s'+1}}(f) \leq ... \leq \hat{C}'_{p_{m'}}(f)$$

$$F_{p_r} > 0, \quad r = 1, ..., s'$$
$$F_{p_r} = 0, \quad r = s' + 1, ..., m'$$

Here marginal costs on used paths are "equilibrated" or equal.

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References

⇒ Dafermos S. and Sparrow F. (1969) The Traffic Assignment Problem for a General Network. *Journal of Research of the National Bureau of Standards*, Vol. 73B, No. 2, April-June 1969, pages 91-118

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For more advanced formulations and associated theory, see Professor Nagurney's Fulbright Network Economics lectures.

http://supernet.som.umass.edu/austria_lectures/fulmain.html