Lecture 3
Cost Structure

Dr. Anna Nagurney

John F. Smith Memorial Professor
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

Cost Structure

Cost is a disutility - Cost is a function of travel time, probability of an accident, scenery of a link.

Assume that all such factors can be lumped together into a disutility.

Both economists and traffic engineers work on determining travel cost functions on the links.

In particular, we consider travel cost functions of a user exercised via links of the network.

Modes of Transportation

www.fredrikmedia.se

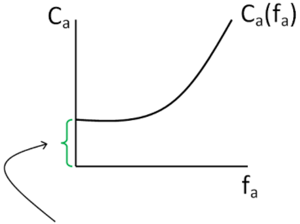
Cost Structure

In the **first generation** model, travel cost of users was assumed constant (depends only on the characteristics of a link and can be determined **a priori** - known as **uncongested** networks.

In the **second generation** model, the networks are **congested**, that is, the user's travel cost depends on the characteristics of the link, but also on the flow on that link.

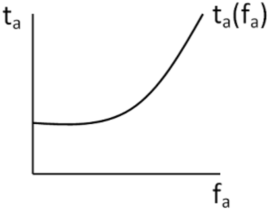
The Simplest Model

The Standard Model



uncongested cost (time)

Often one can substitute cost with time.



Cost Structure

Bureau of Public Roads (BPR) Cost Function

$$c_a = c_a^0 \; [\; 1 \; + \; \alpha \; (\frac{f_a}{t'_a})^\beta \;]$$

where

c_a: travel time on link a
f_a: link flow on link a
c_a^0: free flow travel time
t'_α: "practical capacity" of link a
α, β: model parameters (typically $\alpha = 0.15, \; \beta = 4$)

Modes of Transportation, Marrakech, Morocco

Cost Structure

Suppose now that we have 2 classes of users that perceive cost in different way.

More General Model

$$c_a^1 = c_a^1(f_a^1, f_a^2)$$
$$c_a^2 = c_a^2(f_a^1, f_a^2)$$

Can generalize the 2-class cost structure to k classes or modes.

$*$ But when you make the travel choice you choose paths, not links.

Path Cost Relationship to Link Costs

Let C_p denote the user's or personal travel cost along path p.

$$C_p = C_p(f) = \sum_a c_a(f_a)\delta_{ap},$$

where f is a vector and

$$\delta_{ap} = \begin{cases} 1, & \text{if link } a \text{ is contained in path } p; \\ 0, & \text{otherwise.} \end{cases}$$

Cost Structure

Example: Simplest - Linear

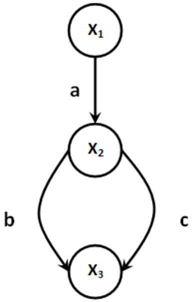
$c_a(f_a) = g_a f_a + h_a$,

$g_a, h_a > 0$ and constant.

g_a is the congestion factor.

$c_a(f_a) = h_a$ - is the uncongested term.

Network Example



$w_1 = (x_1, x_3)$

$p_1 = (a, b)$, $\quad p_2 = (a, c)$

$c_a(f_a) = 10f_a + 5 \quad c_b(f_b) = f_b + 10$

$c_c(f_c) = 5f_c + 5$

Suppose that the travel demand is $d_{w_1} = 10$ and that $F_{p_1} = 5, F_{p_2} = 5.$

What is $C_{p_1} =$? What is $C_{p_2} =$?

Cost Structure

Another type of cost is the **social** or **total** cost.

In the simplest case:

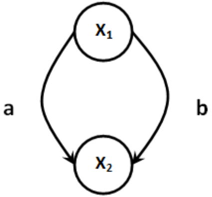
$\hat{c}_a(f_a) = c_a(f_a) \times f_a$

and if c_a is linear, then:

$\hat{c}_a(f_a) = (g_a f_a + h_a) \times f_a = g_a f_a^2 + h_a f_a.$

Hence, if the user cost function on a link is linear, then the total cost is quadratic.

Network Example



$w_1 = (x_1, x_2)$

$c_a(f_a) = 10f_a + 5 \qquad \hat{c}_a(f_a) = 10f_a^2 + 5f_a$

$c_b(f_b) = 4f_b + 10 \qquad \hat{c}_b(f_b) = 4f_b^2 + 10f_b$

Suppose now that

$p_1 = (a), \; p_2 = (b); \; d_{w_1} = 20, \; \textit{and}$

$F_{p_1} = 10, \; F_{p_2} = 10.$

What are the user and total costs on the links a and b?

The Marginal Total Cost

The marginal total cost $\equiv$

$$\frac{\partial \hat{c}_a(f_a)}{\partial f_a}, \quad \text{where } \hat{c}_a(f_a) = c_a(f_a) \times f_a$$

In the uncongested model, the marginal total cost is a constant.

Hence, the marginal total cost in congested networks must be an increasing function of the link flows.

The Total Network Cost

Different ways expressing it.

- $S(f) = \sum_{a} \hat{c}_a(f_a)$
- $S(f) = \sum_{a} c_a(f_a) \times f_a$
- $S(f, F) = \sum_{p} C_p(f) \times F_p$

Cost Structure

For more advanced formulations and associated theory, see Professor Nagurney's Fulbright Network Economics lectures.

http://supernet.som.umass.edu/austria_lectures/fulmain.html