Lecture 11
The Multimodal Model

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Multimodal Transportation

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The Multimodal Model

There are k different modes of transportation using the network: $(1, 2, ..., k)$

Assumptions

- Each mode has its own cost function.
- Each mode contributes to its own and other modes' costs in an individual way.

Multimodal Transportation Networks

Topological characteristics of networks remain the same as the single-modal.

Transportation characteristics now change: The O/D travel demands, path flows, link flows, and path costs now change and become k dimensional vectors.

The Multimodal Model

Travel Costs

c_a^i : user or personal travel cost of mode i on link a

Assumptions

$$
\begin{aligned}
c_a^1 &= c_a^1(f_a^1, \cdots, f_a^k) \\
\vdots \quad & \quad \vdots \\
c_a^k &= c_a^k(f_a^1, \cdots, f_a^k)
\end{aligned}
$$

- extension of the standard model

Travel cost on path p for mode i:

$$C_p^i = C_p^i(f) = \sum_a c_a^i(f_a^1, ..., f_a^k)\delta_{ap}$$

Total link cost of mode i on link a:

$$\hat{c}_a^i = c_a^i(f_a^1, ..., f_a^k) \times f_a^i = \hat{c}_a^i(f_a^1, ..., f_a^k)$$

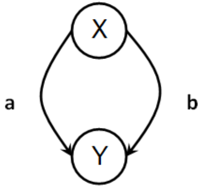
Example of user cost functions

Linear model

$$c_a^i(f_a^1, ..., f_a^k) = \sum_j g_a^{ij} f_a^j + h_a^i$$

$$c_a^1 = g_a^{11} f_a^1 + g_a^{12} f_a^2 + ... + h_a^1$$

Example: 2 modes



$c_a^1 = 10f_a^1 + 5f_a^2 + 10$

$c_a^2 = 5f_a^2 + 4f_a^1 + 5$

$c_b^1 = 6f_b^1 + 4f_b^2 + 10$

$c_b^2 = 3f_b^2 + 2f_b^1 + 15$

$d_{xy}^1 = 10 \qquad d_{xy}^2 = 20$

Conservation of Flow Equations

$d_w^i = \sum_{p \in P_w} F_p^i$, for all O/D pairs w & modes i.

F is feasible if $F_p^i \geq 0$, for all i and p, and the conservation of flow equation above holds.

f_a^i = flow or load on link a induced by mode i.

$$f_a^i = \sum_p F_p^i \, \delta_{ap}$$

$f = (f_a^1, ..., f_a^k)$ is feasible when these equations are satisfied by a feasible path flow pattern.

The Multimodal Model

Travel Demands

With every O/D pair w, we associate a vector travel demand $d_w = (d_w^1, d_w^2, ..., d_w^k)$,

where d_w^i is the travel demand associated with O/D pair w and mode i.

The Multimodal Model

Flows

F_p is now the flow through path p, where F_p is the vector with components $F_p = (F_p^1, F_p^2, ..., F_p^k)$,

and

F_p^i is the flow on path p by mode i.

User-Optimized or Equilibrium Conditions

For each mode i, and every O/D pair w, f is an equilibrium if and only if:

$$C^i_{p_1}(f) = \cdots = C^i_{p_s}(f) = v^i_w \leq C^i_{p_{s+1}}(f) \leq \cdots \leq C^i_{p_m}(f)$$

$\mathrm{F}^i_{p_r}{}^* > 0; r = 1, \cdots, s.$

$\mathrm{F}^i_{p_r}{}^* = 0; r = s+1, \cdots, m.$

where $C^i_p(f) = \sum_a c^i_a(f^1_a, \cdots, f^k_a)\ \delta_{ap}$

The Multimodal Model

The System-Optimized Problem

$$\text{Minimize } S(f) = \sum_i \sum_a \hat{c}_a^i(f_a^1, \cdots, f_a^k)$$

subject to:

$$d_w^i = \sum_{p \in P_w} F_p^i, \text{ for all } i \text{ and } w$$

$$f_a^i = \sum_{p \in P_w} F_p^i \, \delta_{ap}, \text{ for all } i \text{ and } a$$

$$F_p^i \geq 0, \text{ for all } i \text{ and } p.$$

Optimality Conditions for S-O Problem

For each mode i and O/D pair w, f is S-O if and only if:

$$\hat{C'}^{i}_{p_1}(f) = \cdots = \hat{C'}^{i}_{p_{s'}}(f) = \lambda^i_w \leq \hat{C'}^{i}_{p_{s'+1}}(f) \leq \cdots \leq \hat{C'}^{i}_{p_{m'}}(f)$$

$$F^i_{p_r} > 0; r = 1, \cdots, s'$$
$$F^i_{p_r} = 0; r = s'+1, \cdots, m',$$

where

$$\hat{C'}^{i}_{p}(f) = \frac{\partial S}{\partial F^i_p} =^{\text{by chain rule}} \sum_b \sum_j \frac{\partial \hat{c}^j_b}{\partial f^i_b} \delta_{bp}$$

$\hat{C'}^{i}_{p}(f)$: marginal costs

Modes of Transportation

Minnesota Department of Transportation

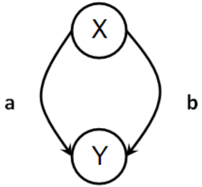
Reduction of Multimodal Networks into a Single Mode Network

Suppose we have a multimodal network. we can construct an equivalent single-modal (but extended cost) network as follows. The network will have a new topology, new travel demands, etc.

We do this by making multiple copies of the network, one copy for each mode of transportation.

Example

(multimodal network)

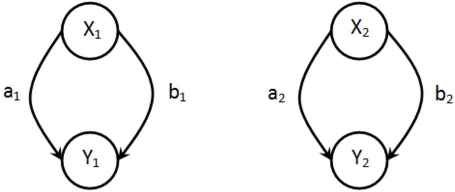


$$d_{xy}^1 = 5, \quad d_{xy}^2 = 10$$

$$c_a^1 = 10f_a^1 + 5f_a^2 + 10 \qquad c_a^2 = 5f_a^2 + 3f_a^1 + 6$$

$$c_b^1 = 5f_b^1 + 4f_b^2 + 3 \qquad c_b^2 = 6f_b^2 + 3f_b^1 + 10$$

Transformation into a single-modal (extended cost) problem:



O/D pair $w_1 = (x_1, y_1)$

$c_{a_1} = 10f_{a_1} + 5f_{a_2} + 10$

$c_{b_1} = 5f_{b_1} + 4f_{b_2} + 3$

$d_{x_1 y_1} = 5$

O/D pair $w_2 = (x_2, y_2)$

$c_{a_2} = 5f_{a_2} + 3_{f\,a_1} + 6$

$c_{b_2} = 6f_{b_2} + 3f_{b_1} + 10$

$d_{x_2 y_2} = 10$

Important Notice

We always denote the U-O, equivalently, transportation network equilibrium solution in either link or path flows with a "*".

References

⇒ Dafermos SC (1972) The traffic assignment problem for multi-class user transportation networks. *Transportation Science* 6: 73-78

Additional Reading

For more advanced formulations and associated theory, see Professor Nagurney's Fulbright Network Economics lectures.

http://supernet.som.umass.edu/austria_lectures/fulmain.html