Lecture 12
The Spatial Price Equilibrium Problem

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The Spatial Price Equilibrium Problem

Parallel to the study of the traffic network equilibrium problem, researchers have studied the spatial price equilibrium problem.

This problem dates to Samuelson (1952), and Takayama and Judge (1971) and is considered the **basic framework** for a variety of applications in energy, agriculture, and interregional/international trade.

New applications have also been proposed in finance.

The Spatial Price Equilibrium Problem

The problem has a structure similar to the TNEP with elastic demands.

In fact, it has been shown by Dafermos & Nagurney (1985), that given any SPE problem, one can construct a "TNEP", which is "isomorphic" and, hence, any SPE problem can be solved as a TNEP.

The Samuelson, Takayama & Judge Model

There are m supply markets and n demand markets involved in the production of a homogeneous commodity.

Let s_i denote the supply of the commodity at supply market i.

Let d_j denote the demand for the commodity at demand market j.

Let Q_{ij} denote the commodity shipment form i to j.

Spatial Price Equilibrium

A set of (s, Q, d) are said to constitute a spatial price equilibrium, if the demand price of the commodity at a demand market is equal to the supply price at the supply market plus the transportation cost **and** there is trade between this pair of supply and demand markets.

If there is no trade between a pair of markets, then the supply price plus transportation cost exceed the demand price.

The Spatial Price Equilibrium Problem

Associated with each supply market i is a supply price function

$$\pi_i = \pi_i(s_i)$$

which is assumed to be increasing.

The Spatial Price Equilibrium Problem

Associated with each demand market j is a demand price function

$$\rho_j = \rho_j(d_j)$$

which is assumed to be decreasing.

Associated with each pair of supply and demand markets (i, j) there is a unit cost of shipping

$$c_{ij} = c_{ij}(Q_{ij})$$

which is assumed to be increasing.

The Spatial Price Equilibrium Problem

Mathematically, this equilibrium state is represented as follows:

For each pair (i, j):

$$\pi_i(s_i) + c_{ij}(Q_{ij}) \begin{cases} = \rho_j(d_j), \text{ if } Q_{ij} > 0 \\ \geq \rho_j(d_j), \text{ if } Q_{ij} = 0 \end{cases}$$

subject to the feasibility conditions:

$$s_i = \sum_j Q_{ij}, \text{ for all supply markets } i$$

$$d_j = \sum_i Q_{ij}, \text{ for all demand markets } j$$

$$Q_{ij} \geq 0, \text{ for all pairs } (i, j).$$

The Spatial Price Equilibrium Problem

These equilibrium conditions have an **equivalent optimization** formulation given by:

$$\text{Minimize} \sum_i \int_0^{s_i} \pi_i(x)dx + \sum_{ij} \int_0^{Q_{ij}} c_{ij}(y)dy - \sum_j \int_0^{d_j} \rho_j(z)dz$$

subject to the previous feasibility conditions.

The Spatial Price Equilibrium Problem

Incorporating the feasibility conditions directly into the above objective function, we obtain:

$$\text{Minimize} \quad \sum_i \int_0^{\sum_j Q_{ij}} \pi_i(x)dx + \sum_{ij} \int_0^{Q_{ij}} c_{ij}(y)dy - \sum_j \int_0^{\sum_i Q_{ij}} \rho_j(z)dz$$

subject to: $Q_{ij} \geq 0$, for all (i, j).

Cargo loading

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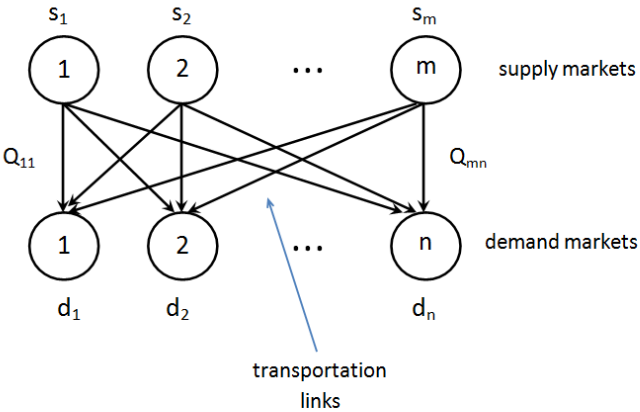
Relationship of the SPEP to TNEP

Obvious network for the SPEP is the bipartite graph where the nodes represent the spatial location of the markets:



The Spatial Price Equilibrium Problem

Indeed, the **Kuhn-Tucker** conditions for this optimization problem are equivalent to the spatial equilibrium conditions.

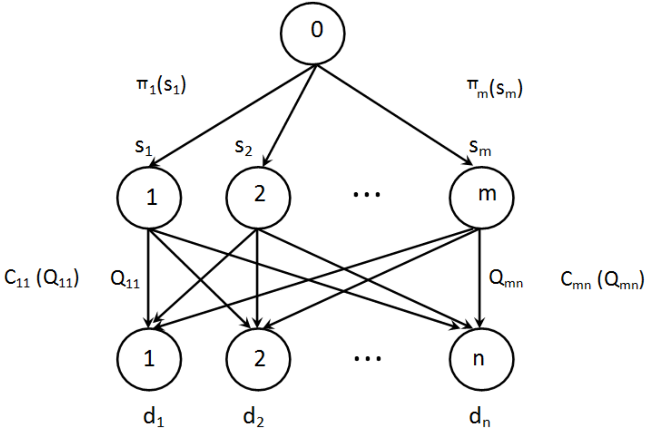
This objective function has been interpreted as a "Social Welfare Function" by many economists.

In the case where the Jacobians of the supply price, demand price, and transportation cost functions

$$[\frac{\partial \pi_i}{\partial s_j}] \quad [\frac{\partial \rho_j}{\partial d_i}] \quad [\frac{\partial c_{ij}}{\partial Q_{kl}}]$$

are symmetric, one also has in optimization reformulation of the SPE conditions.

Construction of the isomorphic traffic network equilibrium problem



The Spatial Price Equilibrium Problem

The O/D pairs are:

$$w_1 = (0, 1), ..., w_n = (0, n).$$

The travel disutilities are:

$$\lambda_{w_1} = \rho_1(d_1), ..., \lambda_{w_n} = \rho_n(d_n).$$

The flow on a path = Q_{ij}.

References

⇒ Samuelson P.A. (1952), Spatial Price Equilibrium and Linear Programming, *The American Economic Review*, Vol. 42, No. 3 (Jun. 1952), pp. 283-303

⇒ Takayama T. and Judge G.G. (1971), *Spatial and Temporal Price and Equilibrium Models*, North Holland, Amsterdam.

⇒ Dafermos S. and Nagurney A. (1985), Isomorphism between spatial price and traffic equilibrium models. Lefschetz Center for Dynamical Systems Report No. 85-17, Division of Applied Mathematics, Brown University, Providence, Rhode Island

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