Traffic Network Equilibrium

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The problem of users of a congested transportation network seeking to determine their travel paths of minimal cost from origins to their respective destinations is a classical network equilibrium problem.

It appears as early as 1920 in the work of Pigou, who considered a two-node, two-link (or path) transportation network, and was further developed by Knight (1924).

The problem has an interpretation as an *economic equilibrium problem* where the demand side corresponds to potential travelers, or consumers, of the network, whereas the supply side is represented by the network itself, with prices corresponding to travel costs. The equilibrium occurs when the number of trips between an origin and a destination equals the travel demand given by the market price, that is, the travel time for the trips.

Wardropian Principles of Traffic

Wardrop (1952) stated the traffic equilibrium conditions through two principles:

First Principle: The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Second Principle: The average journey time is minimal.

The first principle is referred to as *user-optimization* whereas the second is referred to as *system-optimization*.

Beckmann, McGuire, and Winsten (1956) were the first to rigorously formulate these conditions mathematically, as had Samuelson (1952) in the framework of spatial price equilibrium problems in which there were, however, no congestion effects. In particular, Beckmann, McGuire, and Winsten (1956) established the equivalence between the equilibrium conditions and the Kuhn-Tucker conditions of an appropriately constructed optimization problem, under a symmetry assumption on the underlying functions. Hence, in this case, the equilibrium link and path flows could be obtained as the solution of a mathematical programming problem.

Traffic Network Equilibrium Models - Multimodal

Consider now a transportation network. Let a, b, c, etc., denote the links; p, q, etc., the paths. Assume that there are J O/D pairs, with a typical O/D pair denoted by w, and n modes of transportation on the network with typical modes denoted by i, j, etc.

The Link Cost Structure

The flow on a link a generated by mode i is denoted by f_a^i, and the user cost associated with traveling by mode i on link a is denoted by c_a^i. Group the link flows into a column vector $f \in R^{nL}$, where L is the number of links in the network. Group the link costs into a row vector $c \in R^{nL}$. Assume now that the user cost on a link and a particular mode may, in general, depend upon the flows of every mode on every link in the network, that is,

$$c = c(f), \tag{1}$$

where c is a known smooth function.

The Travel Demands and O/D Pair Travel Disutilities

The travel demand of potential users of mode i traveling between O/D pair w is denoted by d^i_w and the travel disutility associated with traveling between this O/D pair using the mode is denoted by λ^i_w. Group the demands into a vector $d \in R^{nJ}$ and the travel disutilities into a vector $\lambda \in R^{nJ}$.

The flow on path p due to mode i is denoted by x^i_p. Group the path flows into a column vector $x \in R^{nQ}$, where Q denotes the number of paths in the network.

Conservation of Flow Equations

The conservation of flows equations are as follows. The demand for a mode and O/D pair must be equal to the sum of the flows of the mode on the paths joining the O/D pair, that is,

$$d^i_w = \sum_{p \in P_w} x^i_p, \quad \forall i, w \tag{2}$$

where P_w denotes the set of paths connecting w.

A nonnegative path flow vector x which satisfies (2) is termed feasible. Moreover, we must have that

$$f^i_a = \sum_p x^i_p \delta_{ap}, \tag{3}$$

that is, that the flow on a link from a mode is equal to the sum of the flows of that mode on all paths that contain that link.

A user traveling on path p using mode i incurs a user (or personal) travel cost C_p^i satisfying

$$C_p^i = \sum_a c_a^i \delta_{ap}, \tag{4}$$

in other words, the cost on a path p due to mode i is equal to the sum of the link costs of links comprising that path and using that mode.

The traffic network equilibrium conditions are given below.

Definition 1 (Traffic Network Equilibrium)

A flow and demand pattern (f^*, d^*) compatible with (2) and (3) is an equilibrium pattern if, once established, no user has any incentive to alter his/her travel arrangements. This state is characterized by the following equilibrium conditions, which must hold for every mode i, every O/D pair w, and every path $p \in P_w$:

$$C_p^i \begin{cases} = \lambda_w^i, & \text{if } {x_p^i}^* > 0 \\ \geq \lambda_w^i, & \text{if } {x_p^i}^* = 0 \end{cases} \tag{5}$$

where λ_w^i is the equilibrium travel disutility associated with the O/D pair w and mode i.

The Elastic Demand Model with Disutility Functions

Assume that there exist travel disutility functions, such that

$$\lambda = \lambda(d), \tag{6}$$

where λ is a known smooth function. That is, let the travel disutility associated with a mode and an O/D pair depend, in general, upon the entire demand pattern.

Let K denote the feasible set defined by

$$K = \{(f, d) \,|\, \exists\, x \geq 0 \,|\, (2)\,, (3)\,\text{hold}\}. \tag{7}$$

The variational inequality formulation of equilibrium conditions (4.5) is given in the next theorem. Assume that λ is a row vector and d is a column vector.

Theorem 1 (Variational Inequality Formulation)

A pair of vectors $(f^*, d^*) \in K$ is an equilibrium pattern if and only if it satisfies the variational inequality problem

$$c(f^*) \cdot (f - f^*) - \lambda(d^*) \cdot (d - d^*) \geq 0, \quad \forall (f, d) \in K. \quad (8)$$

Proof: Note that equilibrium conditions (5) imply that

$$\left[C_p^i(f^*) - \lambda_w^i(d^*)\right] \times \left[x_p^i - {x_p^i}^*\right] \geq 0, \quad (9)$$

for any nonnegative x_p^i. Indeed, if ${x_p^i}^* > 0$, then

$$\left[C_p^i(f^*) - \lambda_w^i(d^*)\right] = 0,$$

and (9) holds; whereas, if ${x_p^i}^* = 0$, then

$$\left[C_p^i(f^*) - \lambda_w^i(d^*)\right] \geq 0,$$

and since $x_p^i \geq 0$, (9) also holds.

Observe that (9) holds for each path $p \in P_w$; hence, one may write

$$\sum_{p \in P_w} \left[C_p^i(f^*) - \lambda_w^i(d^*) \right] \cdot \left[x_p^i - {x_p^i}^* \right] \geq 0, \qquad (10)$$

and, in view of constraint (2), (10) may be rewritten as:

$$\sum_{p \in P_w} C_p^i(f^*) \cdot (x_p^i - {x_p^i}^*) - \lambda_w^i(d^*) \cdot (d_w^i - {d_w^i}^*) \geq 0. \qquad (11)$$

But (11) holds for each mode i and every O/D pair w, hence, one obtains:

$$\sum_{i,w} C_p^i(f^*) \cdot (x_p^i - {x_p^i}^*) - \sum_{i,w} \lambda_w(d^*) \cdot (d_w^i - {d_w^i}^*) \geq 0. \quad (12)$$

In view of (3) and (4), (12) is equivalent to: For $(f^*, d^*) \in K$, induced by a feasible x^*:

$$\sum_{i,a} c_a^i(f^*) \cdot (f_a^i - {f_a^i}^*) - \sum_{i,w} \lambda_w^i(d^*) \cdot (d_w^i - {d_w^i}^*) \cdot (d_w^i - {d_w^i}^*) \geq 0,$$

$$\forall (f, d) \in K, \qquad (13)$$

which, in vector form, yields (8).

We now establish that $(f^*, d^*) \in K$, induced by a feasible x^* and satisfying variational inequality (8) (i.e., (12)), also satisfies equilibrium conditions (5). Fix any mode i, and any path p that joins an O/D pair w. Construct a feasible flow x such that $x^j_q = {x^j_q}^*$ $(j, q) \neq (i, p)$, but $x^i_p \neq {x^i_p}^*$. Then ${d^j_v}^* = d^j_v$, $(j, v) \neq (i, w)$, but $d^i_w = {d^i_w}^* + x^i_p - {x^i_p}^*$. Upon substitution into (12) one obtains

$$C^i_p(f^*) \cdot (x^i_p - {x^i_p}^*) - \lambda^i_w(d^*) \cdot (d^i_w - {d^i_w}^*) \geq 0. \qquad (14)$$

Now, if ${x^i_p}^* > 0$, one may select x^i_p such that $x^i_p > {x^i_p}^*$ or $x^i_p < {x^i_p}^*$, and, consequently, (14) will hold only if $\left[C^i_p(f^*) - \lambda^i_w(d^*)\right] = 0$.

On the other hand, if ${x^i_p}^* = 0$, then $x^i_p \geq {x^i_p}^*$, so that (13) yields

$$C^i_p(f^*) \geq \lambda^i_w(d^*),$$

and the proof is complete.

Qualitative Properties of the Model

Observe that in the above model the feasible set is not compact. Therefore, a condition such as strong monotonicity would guarantee both existence and uniqueness of the equilibrium pattern (f^*, d^*); in other words, if one has that

$$\left[c(f^1) - c(f^2)\right] \cdot \left[f^2 - f^2\right] - \left[\lambda(d^1) - \lambda(d^2)\right] \cdot \left[d^1 - d^2\right]$$

$$\geq \alpha(\|f^1 - f^2\|^2 - \|d^1 - d^2\|^2), \quad \forall (f^1, d^1), (f^2, d^2) \in K, \tag{15}$$

where $\alpha > 0$ is a constant, then there is only one equilibrium pattern.

Condition (15) implies that the user cost function on a link due to a particular mode should depend primarily upon the flow of that mode on that link; similarly, the travel disutility associated with a mode and an O/D pair should depend primarily on that mode and that O/D pair. The link cost functions should be monotonically increasing functions of the flow and the travel disutility functions monotonically decreasing functions of the demand.

The Elastic Demand Model with Demand Functions

We assume that there exist travel demand functions, such that

$$d = d(\lambda) \tag{16}$$

where d is a known smooth function. Assume here that d is a row vector. In this case, the variational inequality formulation of equilibrium conditions (5) is given in the subsequent theorem, whose proof appears in Dafermos and Nagurney (1984a).

Theorem 2 (Variational Inequality Formulation)

Let $\mathcal{M}$ denote the feasible set defined by

$$\mathcal{M} = \{(f, d, \lambda) | \lambda \geq 0, \exists\, x \geq 0 \,|\, (2), (3) \text{ hold}\}. \tag{17}$$

The vector $X^* = (f^*, d^*, \lambda^*) \in \mathcal{M}$ is an equilibrium pattern if and only if it satisfies the variational inequality problem:

$$F(X^*) \cdot (X - X^*) \geq 0, \quad \forall X \in \mathcal{M}, \tag{18}$$

where $F : \mathcal{M} \mapsto R^{n(L+2J)}$ is the function defined by

$$F(f, d, \lambda) = (c(f), -\lambda^T, d - d(\lambda)). \tag{19}$$

Qualitative Properties of the Model

To obtain existence one could impose either a strong monotonicity condition or coercivity condition on the functions c and d. However, strong monotonicity (or coercivity), although reasonable for c, may not be a reasonable assumption for d. The following theorem provides a condition under which the existence of a solution to variational inequality (18) is guaranteed under a weaker condition.

Theorem 3 (Existence)

Let c and d be given continuous functions with the following properties: There exist positive numbers k_1 and k_2 such that

$$c_a^i(f) \geq k_1, \quad \forall a, i \quad \textit{and} \quad f \in \mathcal{M} \tag{20}$$

and

$$d_w^i(\lambda) < k_2, \quad \forall w, \lambda \quad \textit{with} \quad \lambda_w^i \geq k_2. \tag{21}$$

Then (18) has at least one solution.

As in the model with known travel disutility functions, the difficulty of showing existence of a solution for variational inequality (18) is that the feasible set is unbounded. This difficulty can be circumvented as follows. Observe that due to the special structure of the problem, no equilibrium may exist with very large travel demands because such demands would contradict assumption (21), in view of (16). A bounded vector d, in turn, would imply that f is also bounded. This would then imply that $c(f)$ is bounded and, therefore, λ is bounded by virtue of (5) and (1). Consequently, one expects that imposing constants of the type $d \leq \eta$ and $\lambda \leq V$, for η and V sufficiently large, will not affect the set of solutions of (18), while rendering the set compact. We now present a proof through the subsequent two lemmas.

First, fix

$$V > \sum_{f_b^i \leq k_2 J} \max c_a^i(f) \tag{22}$$

and consider the compact, convex set

$$\mathcal{L} = \{(f, d, \lambda) |\, 0 \leq \lambda \leq V; 0 \leq d \leq k_2; \exists x \geq 0 |\, (2), (3) \text{ hold}\}. \tag{23}$$

Consider the variational inequality problem:

Determine $X^* \in \mathcal{L}$, such that

$$F(X^*) \cdot (y - X^*) \geq 0, \quad \forall y \in \mathcal{L}. \tag{24}$$

Since F is continuous and $\mathcal{L}$ is compact, there exists at least one solution, say, $X^* = (f^*, d^*, \lambda^*)$ to (24). The claim is that X^* is actually a solution to the original variational inequality (18).

Lemma 1

If $X^* = (f^*, d^*, \lambda^*)$ *is any solution of variational inequality (24), then*

$$d_w^{i\,*} < k_2, \quad \forall i, w \tag{25}$$

$$\lambda_w^{i\,*} < V, \quad \forall i, w. \tag{26}$$

Lemma 2

Let $X^* = (f^*, d^*, \lambda^*)$ *be a solution of variational inequality (24). Suppose that*

$$d_w^{i\,*} < k_2, \quad \forall w, i \tag{27}$$

$$\lambda_w^{i\,*} < V, \quad \forall w, i. \tag{28}$$

Then X^* *is a solution to the original variational inequality (18).*

Using similar arguments one may establish existence conditions for the model in which travel disutility functions are assumed given, that is, one has the following result.

Theorem 4 (Existence)

Let c and λ be given continuous functions with the following properties: There exist positive numbers k_1 and k_2 such that

$$c_a^i(f) \geq k_1, \quad \forall a, i \quad \textit{and} \quad f \in K$$

and

$$\lambda_w^i(d) < k_1, \quad \forall w, i \quad \textit{and} \quad d \quad \textit{with} \quad d_w^i \geq k_2.$$

Then variational inequality (8) has at least one solution.

The Fixed Demand Model

We now present the fixed demand model is presented is this section. Specifically, it is assumed that the demand d_w^i is now fixed and known for all modes i and all origin/destination pairs w. In this case, the feasible set K would be defined by

$$K = \{f \,|\, \exists\, x \geq 0 \,|\, (2), (3)\, \text{hold}\}. \tag{29}$$

The variational inequality governing equilibrium conditions (5) for this model would be given as in the subsequent theorem. Smith (1979) stated the traffic equilibrium conditions thus whereas Dafermos (1980) identified the formulation as being that of a finite-dimensional variational inequality problem.

Theorem 5 (Variational Inequality Formulation)

A vector $f^* \in K$, is an equilibrium pattern if and only if it satisfies the variational inequality problem

$$c(f^*) \cdot (f - f^*) \geq 0, \quad \forall f \in K. \tag{30}$$

Qualitative Properties

Existence of an equilibrium f^* follows from the standard theory of variational inequalities solely from the assumption that c is continuous, since the feasible set K is now compact.

In the special case where the symmetry condition

$$\left[\frac{\partial c_a^i}{\partial f_b^j} = \frac{\partial c_b^j}{\partial f_a^i} \right], \quad \forall i, j; a, b$$

holds, then the variational inequality problem (30) is equivalent to solving the optimization problem:

$$\text{Minimize}_{f \in K} \sum_{a,i} \int_0^{f_a^i} c_a^i(x) dx. \tag{31}$$

This symmetry assumption, however, is not expected to hold in most applications, and thus the variational inequality problem which is the more general problem formulation is needed. For example, the symmetry condition essentially says that the flow on link b due to mode j should affect the cost of mode i on link a in the same manner that the flow of mode i on link a affects the cost on link b and mode j.

In the case of a single mode problem, the symmetry condition would imply that the cost on link a is affected by the flow on link b in the same manner as the cost on link b is affected by the flow on link a.

In the above framework, with the appropriate construction of the representative network, one can also handle the following situations.

Situation 1: Users of the network have predetermined origins, but are free to select their destinations as well as their travel paths.

Situation 2: Users of the network have predetermined destinations, but they are free to select their origins as well as their travel paths.

Situation 3: Users of the network are free to select their origins, their destinations, as well as their travel paths.

The above situations lead, respectively, to the following network equilibrium problems.

Problem 1: The total number O_u^i of trips produced in each origin node u by each mode (or class) i is given. Determine the O/D travel demands and the equilibrium flow pattern.

Problem 2: The total number D_v^i of trips attracted to each destination node v by each mode i is given. Determine the O/D travel demands and the equilibrium flow pattern.

Problem 3: The total number T^i of trips generated in all origin nodes by all modes i of the network are given, which is equal to the total number of trips attracted to all destinations by each mode. Determine the trip productions O_u^i, the trip attractions D_v^i, the O/D travel demands, and the equilibrium flow pattern.

Here, of course, travel cost should be interpreted liberally. Above we assume that each user of the network, subject to the constraints, chooses his/her origin, and/or destination, and path, so as to minimize his/her travel cost given that all other users have made their choices. The additional factors of attractiveness of the origins and the destinations are taken into account by being incorporated into the model as "travel costs" by a modification of the network through the addition of artificial links with travel cost representing attractiveness.

For example, in Problem 1, we can modify the original network by adding artificial nodes ψ_i, for each mode i, and joining every destination node v of the original network with ψ_i by an artificial link (v, ψ_i). We assume that the travel cost over the artificial links is zero. It is easy to verify that in computing the equilibrium flows according to equilibrium conditions (5) on the expanded network, one can recover the equilibrium flows for the original network. One can make analogous constructions for Problems 2 and 3.

Stability and Sensitivity Analysis

In 1968, Braess presented an example in which the addition of a new link to a network, which resulted in a new path, actually made all the travelers in the network worse off in that the travel cost of all the users was increased. This example, which came to be known as Braess's paradox, generated much interest in addressing questions of stability and sensitivity of traffic network equilibria.

The Braess Paradox

We now present the Braess's paradox example. For easy reference, see the two networks depicted in the Figure.

Assume a network as the first network depicted in the Figure in which there are 4 links: a, b, c, d; 4 nodes: 1,2,3,4; and a single O/D pair $w_1 = (1, 4)$. There are, hence, 2 paths available to travelers between this O/D pair: $p_1 = (a, c)$ and $p_2 = (b, d)$.

The link travel cost functions are:

$$c_a(f_a) = 10f_a \quad c_b(f_b) = f_b + 50$$

$$c_c(f_c) = f_c + 50 \quad c_d(f_d) = 10f_d.$$

Assume a fixed travel demand $d_{w_1} = 6$.

It is easy to verify that the equilibrium path flows are:

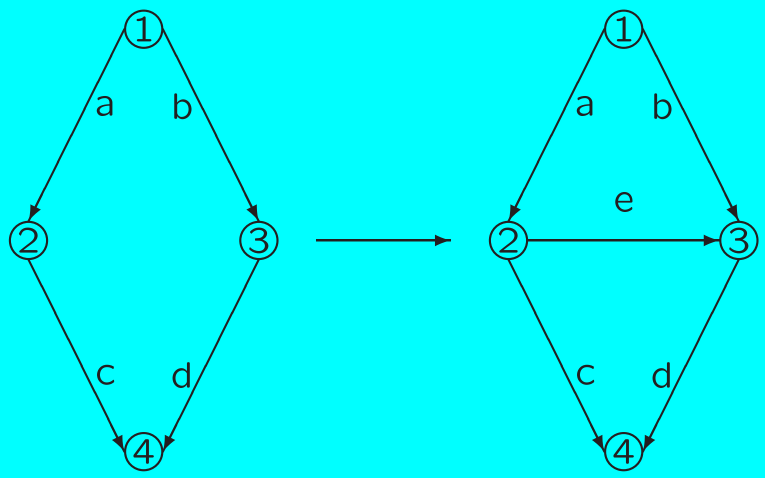
$$x^*_{p_1} = 3 \quad x^*_{p_2} = 3;$$

the equilibrium link flows are:

$$f^*_a = 3 \quad f^*_b = 3 \quad f^*_c = 3 \quad f^*_d = 3;$$

with associated equilibrium path travel costs:

$$C_{p_1} = 83 \quad C_{p_2} = 83.$$



The Braess network example

Assume now that, as depicted in the Figure, a new link "e," joining node 2 to node 3, is added to the original network, with user cost $c_e(f_e) = f_e + 10$. The addition of this link creates a new path $p_3 = (a, e, d)$ that is available to the travelers. Assume that the travel demand d_{w_1} remains at 6 units of flow. Note that the original flow distribution pattern $x_{p_1} = 3$ and $x_{p_2} = 3$ is no longer an equilibrium pattern, since at this level of flow the cost on path p_3, $C_{p_3} = 70$. Hence, users from paths p_1 and p_2 would switch to path p_3.

The equilibrium flow pattern on the new network is:

$$x^*_{p_1} = 2 \quad x^*_{p_2} = 2 \quad x^*_{p_3} = 2;$$

with equilibrium link flows:

$$f^*_a = 4 \quad f^*_b = 2 \quad f^*_c = 2 \quad f^*_e = 2 \quad f^*_d = 4;$$

and with associated equilibrium path travel costs:

$$C_{p_1} = 92 \quad C_{p_2} = 92.$$

Indeed, one can verify that any reallocation of the path flows would yield a higher travel cost on a path.

Note that the travel cost increased for every user of the network from 83 to 92!

We now present the stability results for the models.

Theorem 6

Assume that the strong monotonicity condition (15) is satisfied by the traffic network equilibrium model with known inverse demand functions with constant α. Let (f, d) denote the solution to variational inequality (18) and let (f^*, d^*) denote the solution to the perturbed variational inequality where we denote the perturbations of c and λ by c^* and λ^*, respectively. Then

$$\|((f^*-f),(d-d^*))\| \leq \frac{1}{\alpha}\|((c^*(f^*)-c(f^*)),(\lambda^*(d^*)-\lambda(d^*)))\|. \tag{32}$$

Theorem 7

Assume that $c(f)$ *is strongly monotone with constant* $\bar{\alpha}$ *and that* f *satisfies variational inequality (30). Let* f^* *denote the solution to the perturbed variational inequality with perturbed cost vector* c^**. Then*

$$\|f^* - f\| \leq \frac{1}{\bar{\alpha}}\|c^*(f^*) - c(f^*)\|. \tag{33}$$

In order to attempt to further illuminate paradoxical phenomena in transportation networks, the sensitivity analysis results are presented for the fixed demand model.

Theorem 8

Assume that $f \in K$ satisfies variational inequality (30) and that $f^* \in K$ is the solution to the perturbed variational inequality with perturbed cost vector c^*. Then

$$[c^*(f^*) - c(f)] \cdot [f^* - f] \leq 0. \qquad (34)$$

Inequality (34) may be interpreted as follows: Although an improvement in the cost structure of a network may result in an increase of some of the incurred costs and a decrease in some of the flows, a certain total average cost in the network may be viewed as nonincreasing.

Toll Policies

In this section we describe how tolls, either in the form of path tolls or link tolls, can be imposed in order to make the system-optimizing solution also user-optimizing. Tolls serve as a mechanism for modifying the travel cost as perceived by the individual travelers. We shall show that in the path-toll collection policy there is a degree of freedom that is not available in the link-toll collection policy and how one can take advantage of this added degree of freedom. The analysis is conducted for the traffic network equilibrium model with fixed travel demands.

Recall that the system-optimizing flow pattern is one that minimizes the total travel cost over the entire network, whereas the user-optimized flow pattern has the property that no user has any incentive to make a unilateral decision to alter his/her travel path. One would expect the former pattern to be established when a central authority dictates the paths to be selected, so as to minimize the total cost in the system, and the latter, when travelers are free to select their routes of travel so as to minimize their individual travel cost. The latter solution, however, results in a higher total system cost and, in a sense, is an underutilization of the transportation network. In order to remedy this situation tolls can be applied with the recognition that imposing tolls will not change the travel cost as perceived by society since tolls are not lost.

In particular, in this section it shall be shown how tolls can be collected on a link basis, that is, every member of a class (or mode) on a link will be charged the same toll, irrespective of origin or final destination, or on a path basis, in which every member of a class traveling from an origin to a destination on a particular path will be charged the same toll.

In the link-toll collection policy a toll r_a^i is associated with each link a and mode i. In the path-toll collection policy a toll r_p^i is associated with each path p and mode i.

Of course, even in the link-toll collection policy one may define a "path toll" for class i through the expression

$$r_p^i = \sum_a r_a^i \delta_{ap}. \tag{35}$$

Observe that after the imposition of tolls the travel cost as perceived by society remains $c_a^i(f)$, for all links a and all modes i. The travel cost as perceived by the individual, however, is modified to

$$\bar{C}_p^i = C_p^i(f) + r_p^i, \quad \forall p, i. \tag{36}$$

Consequently, a system-optimizing flow pattern is still defined as before, that is, it is one that solves the problem

$$\text{Minimize}_{f \in K} \sum_{a,i} \hat{c}_a^i(f) \tag{37}$$

where $\hat{c}_a^i(f) = c_a^i(f) \times f_a^i$.

In particular, the solution to (37), under the assumption that each $\hat{c}_a^i(f)$ is convex, is equivalent to the following statement: For every O/D pair w, and every mode i, there exists an ordering of the paths $p \in P_w$, such that

$$\hat{C}_{p_1}^{i'}(f) = \ldots = \hat{C}_{p_{s_i}}^{i'}(f) = \mu_w^i \leq \hat{C}_{p_{s_{i+1}}}^{i'}(f) \leq \ldots \leq \hat{C}_{p_{m_w}}^{i'} \tag{38}$$

$$x_{p_{r_i}}^i > 0, \quad r_i = 1, \ldots, s_i$$

$$x_{p_{r_i}}^i = 0, \quad r_i = s_{i+1}, \ldots, m_w,$$

where m_w denotes the number of paths for O/D pair w. Here we use the notation

$$\hat{C}_p^{i'} = \sum_j \sum_{a,b} \frac{\partial \hat{c}_b^j(f)}{\partial f_a^i} \delta_{ap}. \tag{39}$$

On the other hand, in view of equilibrium conditions (5) one can deduce that the system-optimizing flow pattern x, after the imposition of a toll policy, is at the same time user-optimizing if: For every O/D pair w, every path $p \in P_w$, and every mode i:

$$\bar{C}^i_{p_1}(f) = \ldots = \bar{C}^i_{p_{s_i}}(f) = \bar{\lambda}^i_w \leq \bar{C}^i_{p_{s_i+1}}(f) \leq \ldots \leq \bar{C}^i_{p_{m_w}}(f) \tag{40}$$

$$x^i_{p_{r_i}} > 0, \quad r_i = 1, \ldots, s_i$$

$$x^i_{p_{r_i}} = 0, \quad r_i = s_{i+1}, \ldots, m_w.$$

We now state:

Proposition 1

A toll-collection policy renders a system-optimizing flow pattern user-optimizing if and only if for each mode i, and O/D pair w

$$r^i_{p_1} = \bar{\lambda}^i_w - \bar{C}^i_{p_1}(f)$$

$$\vdots \qquad \qquad \vdots \tag{41}$$

$$r^i_{p_{s_i}} = \bar{\lambda}^i_w - \bar{C}^i_{p_{s_i}}(f)$$

$$r^i_{p_{s_{i+1}}} \geq \bar{\lambda}^i_w - \bar{C}^1_{p_{s_{i+1}}}(f)$$

$$\vdots \qquad \qquad \vdots$$

$$r^i_{p_{m_w}} \geq \bar{\lambda}^i_w - \bar{C}^i_{p_{m_w}}(f). \tag{42}$$

Proof: It is clear that if (38) and (40) are satisfied for the same flow pattern x, then (41) and (42) follow. Conversely, if (41) and (42) are satisfied, then any f that satisfies (38) also satisfies (40).

We now turn to the determination of the link-toll and the path-toll collection policies.

Solution of the Link-Toll Collection Policy

Using (35), (36), and (41) and (42), one reaches the conclusion that the link toll collection policy is determined by

$$r_a^i = \sum_{j,b} \frac{\partial \hat{c}_b^j(f)}{\partial f_a^i} - c_a^i(f) \tag{43}$$

where both the first and the second terms on the right-hand side of expression (43) are evaluated at the system-optimizing solution f.

Usually the link toll pattern constructed above will be the only solution of the link-toll collection problem. There are, however, simple networks in which there may be alternatives.

Hence, to determine the link toll policy, compute the system-optimizing solution. This can be accomplished using a general-purpose convex programming algorithm, an appropriate nonlinear network code, or, in the case of separable linear user cost functions, an equilibration algorithm. Once the system-optimizing solution is established, one then substitutes that flow pattern f into equation (43) to compute the link toll r_a^i for all links a and all modes (or classes) i.

Solution of the Path-Toll Collection Policy

It is obvious from (41) and (42) that one may construct an infinite number of solutions of the path-toll collection problem. For example, one may select, a priori, for each class w, the level of personal travel cost $\bar{\lambda}^i_w$, as well as the values of $r^i_{p_{s_{i+1}}}, \ldots, r^i_{p_{m_w}}$, subject to only constraint (42), and then determine a path toll pattern according to (41). Hence, in this case there is some flexibility in selecting a toll pattern, and one can incorporate additional objectives. Certain possibilities are:

(i) One may wish to ensure that some, if not all, classes of travelers are charged with a nonnegative toll; in other words, no subsidization is allowed for these classes. This can be accomplished by choosing the corresponding $\bar{\lambda}^i_w$ sufficiently large.

(ii) Suppose one wishes a "fair" policy. A possible one would be to ensure that the level of personal travel cost $\bar{\lambda}^i_w$ is equal to the personal travel cost λ^i_w before the imposition of tolls.

We now present a simple example to illustrate how one computes a link toll policy.

An Example

Consider the network depicted the figure in which there are three nodes: 1, 2, 3; three links: a,b,c; and a single O/D pair $w_1 = (1, 3)$. Let path $p_1 = (a, c)$ and path $p_2 = (b, c)$.

Assume, for simplicity, the user cost functions:

$$c_a(f_a) = f_a + 5 \quad c_b(f_b) = 2f_b + 10 \quad c_c(f_c) = f_c + 15,$$

and the travel demand:

$$d_{w_1} = 100.$$

We now turn to the computation of the link toll policy. It is easy to verify that the system-optimizing solution is:

$$x_{p_1} = 67.5 \quad x_{p_2} = 32.5,$$

with associated link load pattern:

$$f_a = 67.5 \quad f_b = 32.5 \quad f_c = 100,$$

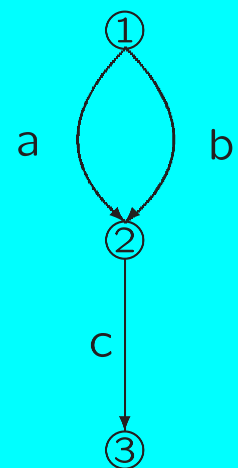
and with marginal path costs:

$$\hat{C}'_{p_1} = \hat{C}'_{p_2} = 355.$$

The link toll policy that renders this system-optimizing flow pattern also user-optimized is given by:

$$r_a = 67.5 \quad r_b = 65 \quad r_c = 100,$$

with the induced user costs $\bar{C}_{p_1} = \bar{C}_{p_2} = 355$.



A link-toll policy example

Computation of Traffic Network Equilibria

We now focus on the computation of traffic network equilibrium problems. In particular, the elastic, multimodal model with known travel disutility functions is considered. The fixed demand model can be viewed as a special case, and the algorithms that will be described here can be readily adapted for the solution of this model as well. Specifically, both the projection method and the relaxation method are presented for this problem domain.

We first present the projection method and then the relaxation method. Assume that the strong monotonicity condition (4.15) is satisfied.

The Projection Method

Step 0: Initialization

Select an initial feasible flow and demand pattern $(f^0, d^0) \in K$. Also, select symmetric, positive definite matrices G and $-M$, where G is an $nL \times nL$ matrix and $-M$ is an $nJ \times nJ$ matrix. Select ρ such that

$$0 < \rho < \min\left[\frac{2\alpha}{\eta}, \frac{2\alpha}{\mu}\right],$$

where α is constant in the strong monotonicity condition, and η and μ are the maximum over K of the maximum of the positive definite symmetric matrices

$$\left[\frac{\partial c}{\partial f}\right]^T G^{-1} \left[\frac{\partial c}{\partial f}\right] \quad \text{and} \quad \left[\frac{\partial \lambda}{\partial d}\right]^T M^{-1} \left[\frac{\partial \lambda}{\partial d}\right].$$

Set $k := 1$.

Step 1: Construction and Computation

Construct

$$h^{k-1} = \rho c(f^{k-1}) - Gf^{k-1} \tag{44}$$

and

$$t^{k-1} = \rho\lambda(d^{k-1}) - Md^{k-1}. \tag{45}$$

Compute the unique user-optimized traffic pattern (f^k, d^k) corresponding to travel cost and disutility functions of the special form:

$$\tilde{c}^{k-1}(f) = Gf + h^{k-1} \tag{46}$$

and

$$\tilde{\lambda}^{k-1}(d) = Md + t^{k-1}. \tag{47}$$

Step 2: Convergence Verification

If $|f^k - f^{k-1}| \leq \epsilon$, with $\epsilon > 0$, a prespecified tolerance, stop; otherwise, set $k := k + 1$, and go to Step 1.

Possibilities for the selection of the matrices G and $-M$ are any diagonal positive definite matrices of appropriate dimensions. One could also set G and M to the diagonal parts of the Jacobian matrices $\left[\frac{\partial c}{\partial f}\right]$ and $\left[\frac{\partial \lambda}{\partial d}\right]$, evaluated at the initial feasible flow pattern. Observe that if one selects diagonal matrices then the above subproblems are decoupled by mode of transportation and each subproblem can be allocated to a distinct processor for computation.

Observe that the projection method constructs a series of symmetric user-optimized problems in which the link user cost functions and the travel disutility functions are linear. Hence, each of these subproblems can be converted into a quadratic programming problem. Moreover, the subproblems can be solved using equilibration algorithms.

We now present the convergence theorem.

Theorem 9

Assume that the strong monotonicity condition (15) holds and that ρ *is constructed as above. Then, for any* $(f^0, d^0) \in K$*, the projection method converges to the solution* (f^*, d^*) *of variational inequality (8).*

The relaxation method for the same model is now presented.

The Relaxation Method

Step 0: Initialization

Select an initial feasible traffic pattern $(f^0, d^0) \in K$. Set $k := 1$.

Step 1: Construction and Computation

Construct new user cost functions:

$$\hat{c}_{(i)} = c_{(i)}(f_{(1)}^{k-1}, \ldots, f_{(i-1)}^{k-1}, f_i, f_{(i+1)}^{k-1}, \ldots, f_{(n)}^{k-1}) \qquad (48)$$

for each mode i, where the subscript i denotes the vector of terms corresponding to mode i.

Construct new travel disutility functions:

$$\hat{\lambda}_{(i)} = \lambda_{(i)}(d_{(1)}^{k-1}, \ldots, d_{(i-1)}^{k-1}, d_i, d_{(i+1)}^{k-1}, \ldots, d_{(n)}^{k-1}) \qquad (49)$$

for each mode i.

Compute the solution to the user-optimized problem with the above travel cost and travel disutility functions for each mode i.

Step 2: Convergence Verification

Same as in Step 2 above in the Projection Method.

Observe that the subproblem encountered at each iteration of the relaxation method will, in general, be a nonlinear problem. Moreover, the above algorithm yields n decoupled subproblems, each of which can also be solved on a distinct processor.

We assume that the variational inequality corresponding to the equilibrium problem with user cost functions (48) and travel disutility functions (49) has a unique solution, which can be computed by a certain algorithm. We now give a condition for convergence of this relaxation method.

Theorem 10

Assume that the functions $\hat{c}_{(i)}$, $\hat{\lambda}_{(i)}$; $i = 1, \ldots, n$, satisfy the monotonicity property:

$$\left[\hat{c}_{(i)}(f'_{(1)}, \ldots, f_{(i)}, \ldots, f'_{(n)}) - \hat{c}_{(i)}(f'_{(1)}, \ldots, \bar{f}_{(i)}, \ldots, f'_{(n)})\right]$$

$$\cdot \left[f_{(i)} - \bar{f}_{(i)}\right]$$

$$-\left[\hat{\lambda}_{(i)}(d'_{(1)}, \ldots, d_{(i)}, \ldots, d'_{(n)}) - \hat{\lambda}_{(i)}(d'_{(1)}, \ldots, \bar{d}_{(i)}, \ldots, d'_{(n)})\right]$$

$$\cdot \left[d_{(i)} - \bar{d}_{(i)}\right] \qquad (50)$$

$$\geq \alpha_1 \|f_{(i)} - \bar{f}_{(i)}\|^2 + \alpha_2 \|d_{(i)} - \bar{d}_{(i)}\|^2,$$

$$\forall (f_{(i)}, d_{(i)}), (\bar{f}_{(i)}, \bar{d}_{(i)}), (f'_{(i)}, d'_{(i)}) \in K,$$

where α_1, α_2 are positive constants.

Also, if there exists a constant γ; $0 < \gamma < 1$, such that

$$\sup\{\sum_{i,j;i\neq j} \|\frac{\partial \hat{c}_{(i)}}{\partial f_{(j)}}\|^2\}^{\frac{1}{2}} \leq \gamma\alpha_1 \qquad (51)$$

$$\sup\{\sum_{i,j;i\neq j} \|\frac{\partial \hat{\lambda}_{(i)}}{\partial d_{(j)}}\|^2\}^{\frac{1}{2}} \leq \gamma\alpha_2 \qquad (52)$$

for all $(f_{(i)}, d_{(i)}) \in K$, then there is a unique solution $(f^*_{(i)}, d^*_{(i)})$; $i = 1, \ldots, n$, to variational inequality (8), and for an arbitrary $(f^0_{(i)}, d^0_{(i)}) \in K$; $i = 1, \ldots, n$;
$(f^k_{(i)}, d^k_{(i)}) \rightarrow (f^*_{(i)}, d^*_{(i)})$; $i = 1, \ldots, n$, as $k \rightarrow \infty$, where (f^*, d^*) satisfies variational inequality (8).

In the case of a single-modal problem, the user cost functions (48) would be separable, that is,

$$\hat{c}_a = c_a(f_1^{k-1}, \ldots, f_a, f_{a+1}^{k-1}, \ldots, f_L^{k-1}), \quad \forall a \tag{53}$$

and the travel disutility functions would also be separable, that is,

$$\hat{\lambda}_w = \lambda_w(d_1^{k-1}, \ldots, d_w, d_w^{k-1}, \ldots, d_J^{k-1}), \quad \forall w, \tag{54}$$

in which case the variational inequality problem at Step 1 would have an equivalent optimization reformulation given by

$$\text{Minimize} \sum_a \int_0^{f_a} \hat{c}_a(x)dx - \sum_w \int_0^{d_w} \hat{\lambda}_w(y)dy \tag{55}$$

subject to $(f, d) \in K$.

The projection method and the relaxation method may also be used to compute the solution to the fixed demand model. In this case, only the user cost functions at each iteration would need to be constructed. Results of numerical testing of these algorithms can be found in Nagurney (1984, 1986). See also Mahmassani and Mouskos (1988).

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