Financial Networks with Intermediation and Transportation Network Equilibria: A Supernetwork Equivalence and Reinterpretation of the Equilibrium Conditions with Computations[1]

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Abstract: In this paper, we consider two distinct classes of network problems – financial networks with intermediation and with electronic transactions and transportation network equilibrium problems, which have been modeled and studied independently. We then prove that the former problem can be reformulated as the latter problem through an appropriately constructed abstract network, i.e., a supernetwork. The established equivalence allows one to then transfer the methodological tools, in particular, algorithms, that have been developed for transportation network equilibria to the financial network domain. In addition, this connection provides us with a novel interpretation of the financial network equilibrium conditions in terms of paths and path flows and a direct existence result. We further show how the theoretical results obtained in this paper can be exploited computationally through

several numerical examples.

Key words: financial networks, transportation network equilibrium, variational inequalities, supernetworks, projected dynamical systems, computation of equilibria

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1. Introduction

Networks have been increasingly used to abstract and mathematically model complex decision-making involving multiple decision-makers in today's linked economies and societies. Applications have ranged from large-scale transportation network equilibrium problems, whose rigorous formulation originates with the book by Beckmann, McGuire, and Winsten (1956) (see also, e. g., Nagurney (1993), Florian and Hearn (1995), and Boyce, Mahmassani, and Nagurney (2005) and the references therein) and telecommunication networks (cf. Resende and Pardalos (2006)) to supply chain networks as well as a spectrum of financial network optimization and equilibrium problems. For an annotated bibliography on network optimization for supply chain and financial engineering problems, see Geunes and Pardalos (2003). For a variety of models and analyses concerning financial engineering, supply chains, as well as electronic commerce, see the volume edited by Pardalos and Tsitsiringos (2002).

The network formalism gaphically captures the structure of distinct problems and applications; enables the identification of possible similarities and differences between networks underlying distinct applications, and allows for the utilization of network-based algorithms for efficient and effective computations. Moreover, the identification of novel (re)formulations of complex, multiple decision-maker problems as network (or supernetwork) problems may lead to new and previously unavailable interpretations of optimality/equilibrium conditions for the entire system. For the history of the term "supernetwork," which is an abstract network, typically, applied to model decision-making on an expanded network, and with origins in transportation science and computer science, see the book by Nagurney and Dong (2002) and Nagurney (2006).

For example, recently, Nagurney (2005) proved that a decentralized supply chain network equilibrium model, due to Nagurney, Dong, and Zhang (2002), in which the decision-makers, in the form of manufacturers, retailers, and consumers, associated with nodes in the distinct tiers of the supply chain network, could be reformulated and solved as a transportation network equilibrium problem with elastic demands originally proposed by Dafermos and Nagurney (1984). This reformulation, established through a supernetwork equivalence, allowed not only for alternative algorithms to be applied for the determination of the supply chain product flows and associated prices, but provided for an entirely new interpretation of the optimality/equilibrium conditions in terms of paths and path flows, which was not apparent in the original supply chain model.

In this paper, we ask the question as to whether financial network problems with intermediation, which possess some of the characteristics of decentralized, multitiered supply chain network problems, can be reformulated as transportation network equilibrium problems? General multitiered financial network problems with intermediation were introduced by Nagurney and Ke (2001) and extended by Nagurney and Ke (2003) to include electronic transactions. Specifically, Nagurney and Ke (2003) considered decision-makers with fixed sources of funds; financial intermediaries, as well as consumers, who were associated with different tiers of the financial network. The decision-makers within one tier of the financial network were allowed to compete with one another in a noncooperative manner. However, decision-makers belonging to different tiers needed to cooperate in order to complete the financial transactions. The authors assumed that the decision-makers with sources of funds (and located at the top tier of the network) and the financial intermediaries (at the middle tier) optimized their own objective functions, which consisted of both net revenue maximization and risk minimization. The consumers, in turn, sought to obtain the financial products such that the price of the financial products charged by the intermediaries or the decision-makers with sources of funds (in the case of direct electronic transactions) plus the respective transaction costs was not greater than the price that consumers were willing to pay for the financial product. The authors assumed that the demand function at each demand market was known, and then formulated the governing equilibrium conditions as a variational inequality (see also Nagurney (1993)). Nagurney and Ke (2003) also provided qualitative analysis as well as an algorithm for computing the equilibrium financial flow and price pattern.

We note that financial systems were first conceptualized as networks in 1758 by Quesnay, where the circular flow of funds in an economy was considered as a graph/network. Thore (1969), in turn, introduced networks and utilized linear programming for the study of systems of linked portfolios (see also Charnes and Cooper (1967)). Thore (1970) then extended the basic network model to handle holdings of financial reserves in the case of uncertainty. The approach made use of two-stage linear programs under uncertainty (cf. Ferguson and

Dantzig (1956) and Dantzig and Madansky (1961)). Storoy, Thore, and Boyer (1975) developed a network model of the interconnection of capital markets and applied decomposition theory of mathematical programming on the computation of equilibrium. Thore (1980) presented network models of linked portfolios with financial intermediation and made the use of decentralization/decomposition theory in the computation. However, the state-of-the-art of that time was not sufficiently developed to allow for the formulation and computation of solutions to general financial network problems with intermediation, which may include competitive behavior in the sense of Nash (1950, 1951), asymmetric functions, etc. Moreover, financial electronic transactions did not even exist in that era. The book by Nagurney and Siokos (1997) provides an overview of a variety of financial network optimization and equilibrium models to that date.

In this paper, we first extend the model of Nagurney and Ke (2003) to the case where the inverse demand (price) functions associated with the demand markets for the financial products are assumed known and given and present underlying behavioral assumptions, along with the variational inequality formulation of the governing equilibrium conditions. We then establish, through the use of a supernetwork equivalence, that this financial network model with intermediation and electronic transactions is equivalent to a transportation network equilibrium model with fixed demands due to Smith (1979) and Dafermos (1980). Hence, in contrast to the transportation network equilibrium reformulation of decentralized supply chain network obtained by Nagurney (2005) which consisted of elastic demands, the transportation network equivalence obtained for the financial network problem with intermediation is one with fixed travel demands associated with the origin/destination pairs. Furthermore, the supernetwork structure of the financial network equilibrium problem with intermediation and electronic transactions is entirely distinct from the one identified for decentralized supply chain networks.

This paper is organized as follows. In Section 2, we present the financial network model with intermediation and electronic transactions. In Section 3, we recall the transportation network equilibrium model with fixed demands due to Smith (1979) and Dafermos (1980). In Section 4, we establish the supernetwork equivalence of the financial network model proposed in Section 2, with a special configuration of the fixed demand transportation/traffic network equilibrium model of Section 3. In particular, we demonstrate that the variational inequality in link form governing the transportation network model coincides with the variational inequality formulation of the financial network model with intermediation and electronic transactions. As a by-product, we also obtain a path flow interpretation of the financial network equilibrium, which is entirely new and not available in the original models. In addition, we obtain an existence result directly by utilizing the supernetwork representation of the financial network equilibrium problem.

In Section 5, we apply the Euler method, proposed by Nagurney and Zhang (1997) for fixed demand traffic network equilibrium problems, and embedded with the exact equilibration algorithm of Dafermos and Sparrow (1969), to compute the equilibrium path flows and link flows for the supernetwork representation of six numerical financial network examples. We conclude the paper with Section 6, in which we summarize the results obtained in this paper and provide suggestions for possible future research. For example, the equivalence established in this paper suggests that international financial network models with intermediation as described in Nagurney and Cruz (2003a, b) can also be transformed and solved as fixed demand traffic network equilibrium problems over appropriately constructed supernetworks.

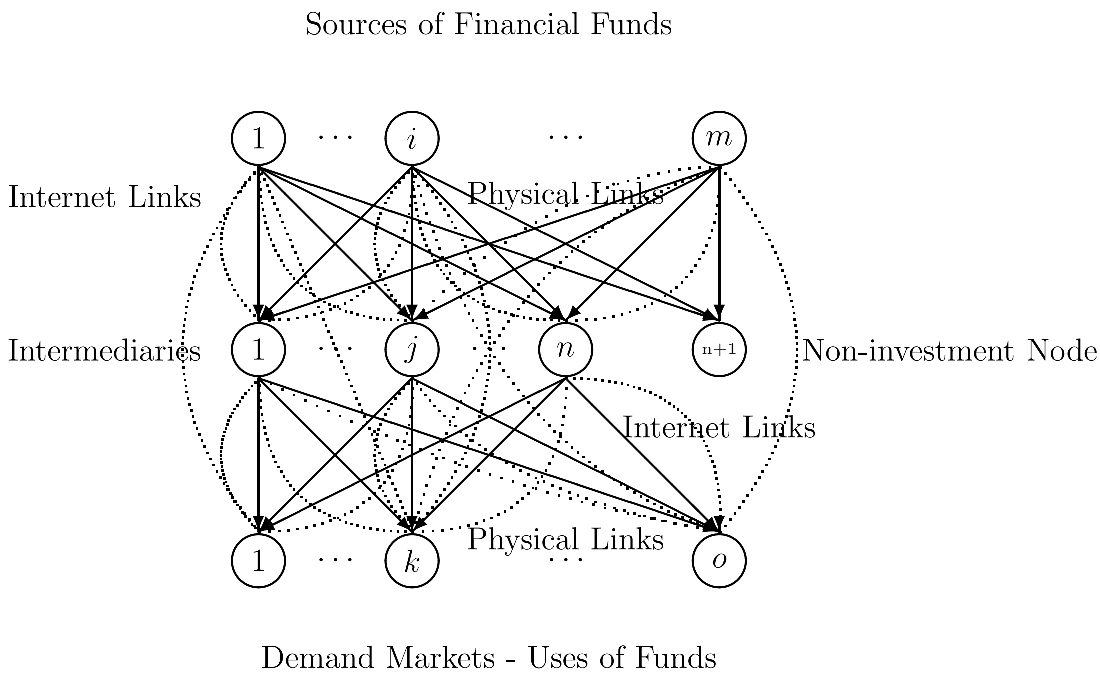

Figure 1: The Structure of the Financial Network with Intermediation and with Electronic Transactions

2. The Financial Network Model with Intermediation and Electronic Transactions

In this Section, we develop the financial network model with intermediation and with electronic transactions in the case of known inverse demand (price) functions associated with the consumers of the financial product at the demand markets. We use the financial network model proposed by Nagurney and Ke (2003) as the foundation.

Similar to the financial network model in Nagurney and Ke (2003), our model consists of m sources of financial funds, n financial intermediaries, and o demand markets, as depicted in Figure 1. In the financial network model, the financial transactions are denoted by the links with the transactions representing electronic transactions delineated by hatched links. The majority of the notation for this model is given in Table 1.

Table 1: Notation for the Financial Network Model

Notation	**Definition**
S	m-dimensional vector of the amounts of funds held by the source agents with component i denoted by S^i
q_i	$(2n+o)$-dimensional vector associated with source agent i; $i=1,\ldots,m$ with components: $\{q_{ijl}; j=1,\ldots,n; l=1,2; q_{ik}; k=1,\ldots,o\}$
q_j	$(2m+2o)$-dimensional vector associated with intermediary j; $j=1,\ldots,n$ with components: $\{q_{ijl}; i=1,\ldots,m; l=1,2; q_{jkl}; k=1,\ldots,o; l=1,2\}$
Q^1	$2mn$-dimensional vector of all the financial transactions/flows for all source agents/intermediaries/modes with component ijl denoted by q_{ijl}
Q^2	mo-dimensional vector of the electronic financial transactions/flows between the sources of funds and the demand markets with component ik denoted by q_{ik}
Q^3	$2no$-dimensional vector of all the financial transactions/flows for all intermediaries/demand markets/modes with component jkl denoted by q_{jkl}
g	n-dimensional vector of the total financial flows received by the intermediaries with component j denoted by g_j, with $g_j \equiv \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}$
γ	n-dimensional vector of shadow prices associated with the intermediaries with component j denoted by γ_j
d	o-dimensional vector of market demands with component k denoted by d_k
$\rho_{3k}(d)$	the inverse demand function at demand market k
V^i	the $(2n+o)\times(2n+o)$ dimensional variance-covariance matrix associated with source agent i
V^j	the $(2m+2o)\times(2m+2o)$ dimensional variance-covariance matrix associated with intermediary j
$c_{ijl}(q_{ijl})$	the transaction cost incurred by source agent i in transacting with intermediary j using mode l with the marginal transaction cost denoted by $\frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}}$
$c_{ik}(q_{ik})$	the transaction cost incurred by source agent i in transacting with demand market k with marginal transaction cost denoted by $\frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}}$
$c_{jkl}(q_{jkl})$	the transaction cost incurred by intermediary j in transacting with demand market k via mode l with marginal transaction cost denoted by $\frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}}$
$c_j(Q^1) \equiv c_j(g)$	conversion/handling cost of intermediary j with marginal handling cost with respect to g_j denoted by $\frac{\partial c_j}{\partial g_j}$ and the marginal handling cost with respect to q_{ijl} denoted by $\frac{\partial c_j(Q^1)}{\partial q_{ijl}}$

Notation	Definition
$\hat{c}_{ijl}(q_{ijl})$	the transaction cost incurred by intermediary j in transacting with source agent i via mode l with the marginal transaction cost denoted by $\frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}}$
$\hat{c}_{jkl}(Q^2, Q^3)$	the unit transaction cost associated with obtaining the product at demand market k from intermediary j via mode l
$\hat{c}_{ik}(Q^2, Q^3)$	the unit transaction cost associated with obtaining the product at demand market k from source agent i

All vectors are assumed to be column vectors. The equilibrium solutions throughout this paper are denoted by *.

The m agents or sources of funds at the top tier of the financial network in Figure 1 seek to determine the optimal allocation of their financial resources transacted either physically or electronically with the intermediaries or electronically with the demand markets. Examples of source agents include: households and businesses. The financial intermediaries, in turn, which can include banks, insurance companies, investment companies, etc., in addition to transacting with the source agents determine how to allocate the incoming financial resources among the distinct uses or financial products associated with the demand markets, which correspond to the nodes at the bottom tier of the financial network in Figure 1. Examples of demand markets are: the markets for real estate loans, household loans, business loans, etc. The transactions between the financial intermediaries and the demand markets can also take place physically or electronically via the Internet.

We denote a typical source agent by i; a typical financial intermediary by j, and a typical demand market by k. The mode of transaction is denoted by l with $l = 1$ denoting the physical mode and with $l = 2$ denoting the electronic mode.

We now describe the behavior of the decision-makers with sources of funds. We then discuss the behavior of the financial intermediaries and, finally, the consumers at the demand markets. Subsequently, we state the financial network equilibrium conditions and derive the variational inequality formulation governing the equilibrium.

The Behavior of the Source Agents

The behavior of the decision-makers with sources of funds, also referred to as source agents in this model, is assumed to be the same as that in Nagurney and Ke (2003). For completeness, it is briefly recalled below.

Since there is the possibility of non-investment allowed, the node $n+1$ in the second tier in Figure 1 represents the "sink" to which the uninvested portion of the financial funds flows from the particular source agent or source node. We then have the following conservation of flow equations:

$$\sum_{j=1}^{n}\sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} \leq S^i, \quad i = 1, \ldots, m, \tag{1}$$

that is, the amount of financial funds available at source agent i and given by S^i cannot exceed the amount transacted physically and electronically with the intermediaries plus the amount transacted electronically with the demand markets. Note that the "slack" associated with constraint (1) for a particular source agent i is given by $q_{i(n+1)}$ and corresponds to the uninvested amount.

Let ρ_{1ijl} denote the price charged by source agent i to intermediary j for a transaction via mode l and, let ρ_{1ik} denote the price charged by source agent i for the electronic transaction with demand market k. The ρ_{1ijl} and ρ_{1ik} are endogenous variables and their equilibrium values ρ^*_{1ijl} and ρ^*_{1ik}; $i = 1, \ldots, m$; $j = 1, \ldots, n$; $l = 1, 2$, $k = 1, \ldots, o$ are determined once the complete financial network model is solved. As noted in the Introduction, we assume that each source agent seeks to maximize his net revenue and to minimize the risk. We assume as in Nagurney and Ke (2001, 2003) that the risk for source agent i is represented by the variance-covariance matrix V^i so that the optimization problem faced by source agent i can be expressed as:

$$\text{Maximize} \quad U^i(q_i) = \sum_{j=1}^{n}\sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} + \sum_{k=1}^{o} \rho^*_{1ik} q_{ik} - \sum_{j=1}^{n}\sum_{l=1}^{2} c_{ijl}(q_{ijl}) - \sum_{k=1}^{o} c_{ik}(q_{ik}) - q_i^T V^i q_i \tag{2}$$

subject to:

$$\sum_{j=1}^{n}\sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} \leq S^i$$

$$q_{ijl} \geq 0, \quad \forall j, l,$$

$$q_{ik} \geq 0, \quad \forall k,$$

$$q_{i(n+1)} \geq 0.$$

The first four terms in the objective function (2) represent the net revenue of source agent i and the last term is the variance of the return of the portfolio, which represents the risk associated with the financial transactions.

We assume that the transaction cost functions for each source agent are continuously differentiable and convex, and that the source agents compete in a noncooperative manner in the sense of Nash (1950, 1951). The optimality conditions for all decision-makers with source of funds simultaneously coincide with the solution of the following variational inequality: determine $(Q^{1*}, Q^{2*}) \in \mathcal{K}^0$ such that:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q_i^* + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q_i^* + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right] \times [q_{ik} - q^*_{ik}] \geq 0, \quad \forall (Q^1, Q^2) \in \mathcal{K}^0, \tag{3}$$

where $V^i_{z_{jl}}$ denotes the z_{jl}-th row of V^i and z_{jl} is defined as the indicator: $z_{jl} = (l-1)n + j$. Similarly, $V^i_{z_{2n+k}}$ denotes the (z_{2n+k})-th row of V^i but with z_{2n+k} defined as the $2n+k$-th row, and the feasible set $\mathcal{K}^0 \equiv \{(Q^1, Q^2)|(Q^1, Q^2) \in R_+^{2mn+mo}$and (1) holds for all $i\}$.

The Behavior of the Financial Intermediaries

The behavior of the intermediaries in the model is identical to that in Nagurney and Ke (2003). For completeness and easy reference, it is recalled below.

Let the endogenous variable ρ_{2jkl} denote the product price charged by intermediary j with ρ^*_{2jkl} denoting the equilibrium price, where $j = 1, \ldots, n$; $k = 1, \ldots, o$, and $l = 1, 2$. We assume that each financial intermediary also seeks to maximize his net revenue while minimizing the risk. Note that a financial intermediary, by definition, may transact either with decision-makers in the top tier of the financial network as well as with consumers associated with the demand markets in the bottom tier. Noting the conversion/handling cost as well as the various transaction costs faced by a financial intermediary and recalling that the variance-covariance matrix associated with financial intermediary j is given by V^j (cf. Table 1), we have that the financial intermediary is faced with the following optimization problem:

$$\text{Maximize} \quad U^j(q_j) = \sum_{k=1}^{o}\sum_{l=1}^{2}\rho^*_{2jkl}q_{jkl} - c_j(Q^1) - \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{c}_{ijl}(q_{ijl}) - \sum_{k=1}^{o}\sum_{l=1}^{2}c_{jkl}(q_{jkl})$$

$$-\sum_{i=1}^{m}\sum_{l=1}^{2}\rho^*_{1ijl}q_{ijl} - q_j^T V^j q_j \tag{4}$$

subject to:

$$\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl} \le \sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}, \tag{5}$$

$$q_{ijl} \ge 0, \quad \forall i, l,$$

$$q_{jkl} \ge 0, \quad \forall k, l.$$

The first five terms in the objective function (4) denote the net revenue, whereas the last term is the variance of the return of the financial allocations, which represents the risk. Constraint (5) guarantees that an intermediary cannot reallocate more of its financial funds among the demand markets then it has available.

Let γ_j be the Lagrangian multiplier associated with constraint (5) for intermediary j. We assume that the cost functions are continuously differentiable and convex, and that the intermediaries compete in a noncooperative manner. Hence, the optimality conditions for all intermediaries simultaneously can be expressed as the following variational inequality: determine $(Q^{1*}, Q^{3*}, \gamma^*) \in R_+^{2mn+2no+n}$ satisfying:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^j_{z_{il}}\cdot q^*_j + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma^*_j\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}}\cdot q^*_j + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} - \rho^*_{2jkl} + \gamma^*_j\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q^*_{ijl} - \sum_{k=1}^{o}\sum_{l=1}^{2}q^*_{jkl}\right] \times \left[\gamma_j - \gamma^*_j\right] \ge 0, \quad \forall (Q^1, Q^3, \gamma) \in R_+^{2mn+2no+o}, \tag{6}$$

where $V^j_{z_{il}}$ denotes the z_{il}-th row of V^j where z_{il} is defined as the indictor: $z_{il} = (l-1)m+i$. Similarly, $V^j_{z_{kl}}$ denotes the z_{kl}-th row of V^j and z_{kl} is the indicator: $z_{kl} = 2m+(l-1)o+k$.

Additional background on risk management in finance can be found in Nagurney and Siokos (1997); see also the book by Rustem and Howe (2002).

The Consumers at the Demand Markets and the Equilibrium Conditions

Unlike the models of Nagurney and Ke (2001, 2003), we now assume, as given, the inverse demand functions $\rho_{3k}(d)$; $k = 1, \ldots, o$, associated with the demand markets at the bottom tier of the financial network. Recall that the demand markets correspond to distinct financial products. Of course, if the demand functions are invertible, then one may obtain the price functions simply by inversion.

The following conservation of flow equations must hold:

$$d_k = \sum_{j=1}^{n} \sum_{l=1}^{2} q_{jkl} + \sum_{i=1}^{m} q_{ik}, \quad k = 1, \ldots, o. \tag{7}$$

Equations (7) state that the demand for the financial product at each demand market is equal to the financial transactions from the intermediaries to that demand market plus those from the source agents.

The equilibrium condition for the consumers at demand market k are as follows: for each intermediary j; $j = 1, \ldots, n$ and mode of transaction l; $l = 1, 2$:

$$\rho^*_{2jkl} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) \begin{cases} = \rho_{3k}(d^*), & \text{if} \quad q^*_{jkl} > 0 \\ \geq \rho_{3k}(d^*), & \text{if} \quad q^*_{jkl} = 0. \end{cases} \tag{8}$$

In addition, we must have that, in equilibrium, for each source of funds i; $i = 1, \ldots, m$:

$$\rho^*_{1ik} + \hat{c}_{ik}(Q^{2*}, Q^{3*}) \begin{cases} = \rho_{3k}(d^*), & \text{if} \quad q^*_{ik} > 0 \\ \geq \rho_{3k}(d^*), & \text{if} \quad q^*_{ik} = 0. \end{cases} \tag{9}$$

Condition (8) states that, in equilibrium, if consumers at demand market k purchase the product from intermediary j via mode l, then the price the consumers pay is exactly equal to the price charged by the intermediary plus the unit transaction cost via that mode. However, if the sum of price charged by the intermediary and the unit transaction cost is greater than the price the consumers are willing to pay at the demand market, there will be no transaction between this intermediary/demand market pair via that mode. Condition (9) states the analogue but for the case of electronic transactions with the source agents.

In equilibrium, conditions (8) and (9) must hold for all demand markets. We can also express these equilibrium conditions using the following variational inequality: determine $(Q^{2*}, Q^{3*}, d^*) \in \mathcal{K}^1$, such that

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\rho^*_{2jkl} + \hat{c}_{jkl}(Q^{2*}, Q^{3*})\right] \times \left[q_{jkl} - q^*_{jkl}\right] + \sum_{i=1}^{m}\sum_{k=1}^{o}\left[\rho^*_{1ik} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q^*_{ik}]$$

$$- \sum_{k=1}^{o} \rho_{3k}(d^*) \times [d_k - d^*_k] \geq 0, \quad \forall (Q^2, Q^3, d) \in \mathcal{K}^1, \tag{10}$$

where $\mathcal{K}^1 \equiv \{(Q^2, Q^3, d) | (Q^2, Q^3, d) \in R^{2no+mo+o}_{+}$ and (7) holds.$\}$

The Equilibrium Conditions for Financial Network with Electronic Transactions

In equilibrium, the optimality conditions for all decision-makers with source of funds, the optimality conditions for all the intermediaries, and the equilibrium conditions for all the demand markets must be simultaneously satisfied so that no decision-maker has incentive to alter his or her transactions. We now formally state the equilibrium condition for the entire financial network with intermediation and electronic transactions as follows.

Definition 1: Financial Network Equilibrium with Intermediation and with Electronic Transactions

The equilibrium state of the financial network with intermediation is one where the financial flows between tiers coincide and the financial flows and prices satisfy the sum of conditions (3), (6), and (10).

We now define the feasible set:

$$\mathcal{K}^2 \equiv \{(Q^1, Q^2, Q^3, \gamma, d) | (Q^1, Q^2, Q^3, \gamma, d) \in R^{m+2mn+2no+mo+o}_{+} \text{ and (1) and (7) hold}\}$$

and state the following theorem.

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the financial network model with intermediation are equivalent to the solution to the variational inequality problem given by: determine $(Q^{1*}, Q^{2*}, Q^{3*}, \gamma^*, d^*) \in \mathcal{K}^2$ *satisfying:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q^*_i + 2V^j_{z_{il}} \cdot q^*_j + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma^*_j\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q^*_i + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q^*_{ik}]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}} \cdot q^*_j + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) + \gamma^*_j\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q^*_{ijl} - \sum_{k=1}^{n}\sum_{l=1}^{2} q^*_{jkl}\right] \times \left[\gamma_j - \gamma^*_j\right] - \sum_{k=1}^{o} \rho_{3k}(d^*) \times [d_k - d^*_k] \geq 0,$$

$$\forall (Q^1, Q^2, Q^3, \gamma, d) \in \mathcal{K}^2. \tag{11}$$

Proof: We first prove that an equilibrium according to Definition 1 satisfies variational inequality (11). Summation of (3), (6), and (10), after algebraic simplifications, yields (11).

We now prove the converse, that is, that a solution to variational inequality (11) satisfies the sum of conditions (3), (6), and (10), and is, therefore, a financial network equilibrium pattern. First, we add the term: $-\rho^*_{1ijl} + \rho^*_{1ijl}$ to the term in the first set of brackets in (11). Then, we add the term: $-\rho^*_{1ik} + \rho^*_{1ik}$ to the term before the second multiplication sign, and, finally, we add the term $-\rho^*_{2jkl} + \rho^*_{2jkl}$ to the term preceding the third multiplication sign in (11). Note that these terms are identically equal to zero and do not change the variational inequality. We obtain, hence, the following inequality:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q^*_i + 2V^j_{z_{il}} \cdot q^*_j + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma^*_j - \rho^*_{1ijl} + \rho^*_{1ijl}\right]$$

$$\times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}}\cdot q_i^* + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*},Q^{3*}) - \rho_{1ik}^* + \rho_{1ik}^*\right]\times[q_{ik}-q_{ik}^*]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}}\cdot q_j^* + \frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*},Q^{3*}) + \gamma_j^* - \rho_{2jkl}^* + \rho_{2jkl}^*\right]\times\left[q_{jkl}-q_{jkl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^* - \sum_{k=1}^{n}\sum_{l=1}^{2}q_{jkl}^*\right]\times\left[\gamma_j-\gamma_j^*\right] - \sum_{k=1}^{o}\rho_{3k}(d^*)\times[d_k-d_k^*]\geq 0,$$

$$\forall (Q^1,Q^2,Q^3,\gamma,d)\in\mathcal{K}^2, \tag{12}$$

which, can be rewritten as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o}\left[2V^i_{z_{jl}}\cdot q_i^* + \frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \rho_{1ijl}^*\right]\times\left[q_{ijl}-q_{ijl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}}\cdot q_i^* + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} - \rho_{1ik}^*\right]\times[q_{ik}-q_{ik}^*]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^j_{z_{il}}\cdot q_j^* + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho_{1ijl}^* + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \gamma_j^*\right]\times\left[q_{ijl}-q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}}\cdot q_j^* + \frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}} - \rho_{2jkl}^* + \gamma_j^*\right]\times\left[q_{jkl}-q_{jkl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^* - \sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl}^*\right]\times\left[\gamma_j-\gamma_j^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\rho_{2jkl}^* + \hat{c}_{jkl}(Q^{2*},Q^{3*})\right]\times\left[q_{jkl}-q_{jkl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\rho_{1ik}^* + \hat{c}_{ik}(Q^{2*},Q^{3*})\right]\times[q_{ik}-q_{ik}^*] - \sum_{k=1}^{o}\rho_{3k}(d^*)\times[d_k-d_k^*]\geq 0,$$

$$\forall (Q^1,Q^2,Q^3,\gamma,d)\in\mathcal{K}^2. \tag{13}$$

Obviously, the solution to inequality (13) satisfies the sum of the conditions (3), (6), and (10). The proof is complete. □

The variables in the variational inequality problem (11) are: the uninvested portion of funds, Q^0; the financial flows from the source agents to the intermediaries, Q^1; the direct financial flows via electronic transaction from the source agents to the demand markets, Q^2; the financial flows from the intermediaries to the demand markets, Q^3; the shadow prices associated with handling the product by the intermediaries, γ, and the demands at demand markets d. The solution to the variational inequality problem (11), $(Q^{0^*}, Q^{1^*}, Q^{2^*}, Q^{3^*}, \gamma^*, d^*)$, coincides with the equilibrium financial flow and price pattern according to Definition 1.

Variational inequality (11) is distinct from the variational inequality derived in Nagurney and Ke (2003) in which the demand functions at the markets were assumed known and given.

We now state and prove two corollaries and derive variational inequality formulations alternative to (11) which we will utilize in Section 4 to construct the supernetwork equivalence of the financial network equilibrium model to a properly configured transportation network equilibrium model with fixed demand.

Corollary 1

The market for the financial flows clears for each intermediary at the equilibrium.

Proof: We will show that in equilibrium, for each intermediary, the total amount of financial inflows is equal to the total amount of financial outflows, that is, $\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^* = \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl}^*$.

From the fourth term of (11), we can clearly see that if $\gamma_j^* > 0$, then $\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^* = \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl}^*$ must hold.

Let us now consider the case where $\gamma_j^* = 0$ for some intermediary j. Since we have assumed that the transaction cost functions and handling cost functions are convex, it is reasonable to further assume that, in equilibrium, either the marginal transaction costs or the marginal handling cost for each source agent/intermediary/mode combination is strictly positive. Then, we have that $2V_{z_{jl}}^i \cdot q_i^* + 2V_{z_{il}}^j \cdot q_j^* + \frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} > 0$, which implies that $q_{ijl}^* = 0$, and this holds for all i, j, l. It follows from the fourth term of (11) that $\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^* = \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl}^* = 0$.

Therefore, the market of financial flows clears at each intermediary in equilibrium. The proof is complete. $\square$

Since we are interested in determining the equilibrium flow and price pattern, it is reasonable and convenient to convert the constraint (5) into the following equality form:

$$\sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl} = \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}, \tag{14}$$

and to define the feasible set

$$\mathcal{K}^3 \equiv \{(Q^1, Q^2, Q^3, d) | (Q^1, Q^2, Q^3, d) \in R_+^{m+2mn+mo+2no+o} \text{ and } (1), (7), \text{ and } (14) \text{ hold}\}.$$

In addition, for notational convenience, we let

$$g_j \equiv \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}, \quad i = 1, \ldots, m. \tag{15}$$

As defined in Table 1, the conversion/handling cost of intermediary j, c_j, is a function of the total financial inflows of intermediary j:

$$c_j(Q^1) \equiv c_j(g). \tag{16}$$

Hence, its marginal cost with respect to q_{ijl} is equal to the marginal cost with respect to g_j.

$$\frac{\partial c_j(Q^1)}{\partial q_{ijl}} \equiv \frac{\partial c_j(g)}{\partial g_j}. \tag{17}$$

Corollary 2 then follows immediately:

Corollary 2

A solution $(Q^{1*}, Q^{2*}, Q^{3*}, d^*) \in \mathcal{K}^3$ to the variational inequality problem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q^*_i + 2V^j_{z_{il}} \cdot q^*_j + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q^*_i + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q^*_{ik}]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}} \cdot q^*_j + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*})\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$

$$-\sum_{k=1}^{o} \rho_{3k}(d^*) \times [d_k - d_k^*] \geq 0, \quad \forall (Q^1, Q^2, Q^3, d) \in \mathcal{K}^3; \tag{18a}$$

equivalently, a solution $(Q^{1*}, Q^{2*}, Q^{3*}, g^*, d^*) \in \mathcal{K}^4$ *to*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{jl}}^{i} \cdot q_i^* + 2V_{z_{il}}^{j} \cdot q_j^* + \frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}\right] \times \left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V_{z_{2n+k}}^{i} \cdot q_i^* + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q_{ik}^*]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V_{z_{kl}}^{j} \cdot q_j^* + \frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*})\right] \times \left[q_{jkl} - q_{jkl}^*\right]$$

$$+\sum_{j=1}^{n}\frac{\partial c_j(g^*)}{\partial g_j} \times \left[g_j - g_j^*\right] - \sum_{k=1}^{o} \rho_{3k}(d^*) \times [d_k - d_k^*] \geq 0, \quad \forall (Q^1, Q^2, Q^3, g, d) \in \mathcal{K}^4, \tag{18b}$$

where

$$\mathcal{K}^4 \equiv \{(Q^1, Q^2, Q^3, g, d) | (Q^1, Q^2, Q^3, g, d) \in R_+^{m+2mn+mo+2no+n+o}$$

$$\textit{and } (1),\ (7),\ (14),\ \textit{and } (15)\ \textit{hold}\}$$

satisfies variational inequality (11).

Proof: We prove the result by contradiction. In particular, we show that if a financial flow pattern is not a solution of (11), then it is not a solution of (18a). Thus, we assume that there exists some $(Q^1, Q^2, Q^3, d) \in \mathcal{K}^3$ and $\gamma \in R_+^n$ such that the left-hand side of (11) is less than zero, which implies that:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{jl}}^{i} \cdot q_i^* + 2V_{z_{il}}^{j} \cdot q_j^* + \frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}\right] \times \left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V_{z_{2n+k}}^{i} \cdot q_i^* + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q_{ik}^*]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V_{z_{kl}}^{j} \cdot q_j^* + \frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*})\right] \times \left[q_{jkl} - q_{jkl}^*\right]$$

$$-\sum_{k=1}^{o} \rho_{3k}(d^*) \times [d_k - d_k^*]$$

$$< \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{2} \gamma_j^* \times \left[q_{ijl} - q_{ijl}^*\right] + \sum_{j=1}^{n} \sum_{k=1}^{o} \sum_{l=1}^{2} \gamma_j^* \times \left[q_{jkl} - q_{jkl}^*\right]$$

$$+ \sum_{j=1}^{n} \left[\sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl}^* - \sum_{k=1}^{n} \sum_{l=1}^{2} q_{jkl}^*\right] \times \left[\gamma_j - \gamma_j^*\right]. \quad (19)$$

After the use of Corollary 1 and algebraic simplification, we obtain the result that the right-hand side of (19) is equal to zero. Hence, (18a) cannot hold, and the conclusion follows.

To show that (18a) and (18b) are equivalent, we utilize (15) and (17), which upon substitution into (18a), after algebraic simplification, and the use of (7), yields (18b). □

3. The Transportation Network Equilibrium Model with Fixed Demands

In this Section, we review a transportation network equilibrium model with fixed demands due to Smith (1979) and Dafermos (1980). In the model, travelers, also referred to as users, seek to determine their minimal costs of travel from origins to destinations subject to the travel demands associated with the origin/destination pairs of nodes. The governing behavioral concept is that is user-optimization, a term coined by Dafermos and Sparrow (1969). The concept is also known as Wardrop's (cf. Wardrop (1952)) first principle (in contrast to the second principle which assumes a single controller that can route the traffic on the network and is now commonly referred to as system-optimization). Hence, users/travelers are assumed to compete in a unilateral fashion. As noted in the Introduction, the first rigorous formulation of transportation network equilibrium is due to Beckmann, McGuire, and Winsten (1956), who actually proposed elastic demand models and demonstrated that, in the case of symmetric functions, the governing equilibrium conditions coincided with the Kuhn-Tucker conditions of an appropriately constructed optimization problem. The work of Smith (1979) and Dafermos (1980) showed that transportation network equilibrium problems in the case of asymmetric functions, for which such optimization reformulations of the governing equilibrium conditions did not exist, could be formulated, qualitatively analyzed, and solved as variational inequality problems. For additional background on transportation network equilibrium models in a variational inequality framework, see the books by Nagurney (1993) and Patriksson (1994).

We consider a network G with the set of links L consisting of K elements; the set of paths P consisting of Q elements, and the set of origin/destination (O/D) pairs W with Z elements. We also denote the set of paths connecting O/D pair w by P_w. In this model, links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w_1, w_2, etc.

We denote the nonnegative flow on path p by x_p and the flow on link a by f_a. The user (travel) cost on a path p is denoted by C_p and the user (travel) cost on a link a by c_a. We denote the fixed travel demand associated with O/D pair w by d_w and the travel disutility by λ_w.

The travel demand associated with each O/D pair must satisfy the following equation:

$$d_w = \sum_{p \in P_w} x_p, \quad \forall w, \tag{20}$$

that is, the sum of the path flows on paths connecting each O/D pair must be equal to the travel demand associated with that O/D pair. Note that in the fixed demand model described here the travel demands are assumed known.

The link flows and the path flows, in turn, are related by the following conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{21}$$

where $\delta_{ap} = 1$ if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Note that (20) states that the flow on a link is equal to the sum of the flows on the paths that contain that link.

Since we will provide variational inequality formulations of the governing transportation network equilibrium conditions in path flows as well as in link flows, we need to define the corresponding definitions of the underlying feasible sets. We define the feasible set $\mathcal{K}^5 \equiv \{x | x \geq 0, \text{ and } (20) \text{ holds}\}$ as well as the feasible set $\mathcal{K}^6 \equiv \{f | \exists x \geq 0, \text{ and satisfying } (20), \text{ and } (21)\}$.

The (user) cost on a path is equal to the sum of the (user) costs on links which comprise the path, that is,

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P. \tag{22}$$

The link cost c_a, in turn, in general, may depend on the vector of link flows, denoted by f, where we may write

$$c_a = c_a(f), \quad \forall a \in L. \tag{23}$$

For a fixed demand transportation network, a path flow pattern is said to be an equilibrium path flow pattern and denoted by x^*, if, once established, no user has any incentive to alter his travel choices. This statement can be formally expressed by the following equilibrium conditions which must hold for every O/D pair $w \in W$ and every path $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0. \end{cases} \tag{24}$$

Condition (24) states that for each O/D pair, all used paths have equal and minimal costs. As established in Smith (1979) and Dafermos (1980) (for further background, see Nagurney (1993)), these equilibrium conditions can be expressed by the following variational inequality in path flows: determine $x^* \in \mathcal{K}^5$ such that

$$\sum_{w\in W}\sum_{p\in P_w} C_p(x^*) \times \left[x_p - x_p^*\right] \geq 0, \quad \forall x \in \mathcal{K}^5. \tag{25}$$

We also provide the equivalent variational inequality formulation in link flows due to Smith (1979) and Dafermos (1980).

Theorem 2

A link flow pattern is a transportation network equilibrium if and only if it satisfies the variational inequality problem: determine $f^* \in \mathcal{K}^6$ satisfying

$$\sum_{a\in L} c_a(f^*) \times [f_a - f_a^*] \geq 0, \quad \forall f \in \mathcal{K}^6. \tag{26}$$

In the next Section, we will establish the supernetwork equivalence of the financial network model introduced in Section 2 with a properly configured fixed demand traffic network model as just outlined. In particular, we will show that the link form variational inequality (26) coincides with variational inequality (18b) for the supernetwork representation of the financial network model with intermediation.

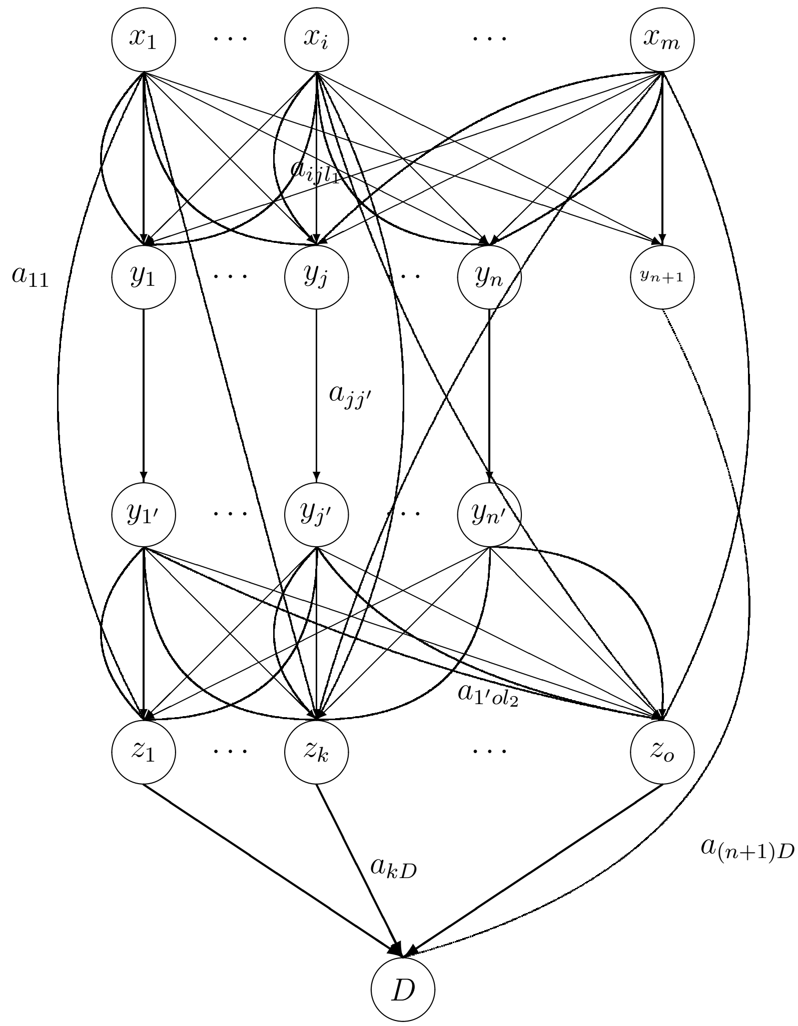

Figure 2: The G_S Supernetwork Representation of Financial Network Equilibrium

4. Supernetwork Equivalence of the Financial Network Equilibrium Model

In this Section, we show that the financial network equilibrium model with intermediation presented in Section 2 is isomorphic to a properly configured transportation network equilibrium model through the construction of a supernetwork equivalence of the former. We then illustrate how this result can be exploited theoretically by providing a new interpretation of the financial equilibrium conditions in terms of paths and path flows. We also demonstrate through several numerical examples in Section 5 how algorithms developed for the solution of transportation network equilibrium problems with fixed demands can be applied to compute the equilibrium solution to the financial network problem.

Consider a financial network problem with intermediation and electronic transactions as described in Section 2 consisting of m source agents; n financial intermediaries, and o demand markets denoting distinct financial products. The supernetwork, G_S, of the isomorphic transportation network equilibrium problem is depicted in Figure 2 and is constructed as follows. It consists of five tiers of nodes with the origin nodes at the top (or first) tier and the destination node at the fifth or bottom tier. Specifically, G_S consists of m origin nodes at the top tier, denoted, respectively, as: x_1, x_2, ..., x_m and a single destination node at the bottom tier denoted by D. There are m O/D pairs in G_S denoted by $w_i = (x_i, D)$; $i = 1, \ldots, m$. Each top tiered node x_i is connected to each second tiered node y_j; $j = 1, \ldots, n$ by two parallel links, and also connected to each fourth tiered node z_j; $j = 1, \ldots, n$ by a single link. There is also a link connecting each node x_i to node y_{n+1}. Every second tiered node y_j; $j = 1, \ldots, n$ except for y_{n+1} is connected to the corresponding third tiered node $y_{j'}$; $j' = 1', \ldots, n'$ by a single link. Hence, we have that $n' \equiv n$. Node y_{n+1} is connected to the destination node D by a single link. Each third tiered node $y_{j'}$, in turn, is connected to each fourth tiered node z_k by two parallel links. Finally, each fourth tiered node z_k is connected to the destination node D by a single link.

Hence, in G_S, there are $m + 2n + o + 2$ nodes and $K = 2mn + m + mo + n + 2no + o + 1$ links. We now define the link notation as well as the link flows. Let $a_{i(n+1)}$ denote the link from x_i to y_{n+1} with associated link flow $f_{a_{i(n+1)}}$ for $i = 1, \ldots, m$. Let a_{ijl_1} denote the l_1^{th} link from node x_i to node y_j with associated link flow $f_{a_{ijl_1}}$ for $i = 1, ..., m$, $j = 1, .., n$, and $l_1 = 1, 2$. Let a_{ik} denote the link from node x_i to node z_k with associated link flow $f_{a_{ik}}$ for $i = 1, ..., m$ and $k = 1, .., o$. Let $a_{jj'}$ denote the link from node y_j to node $y_{j'}$ with associated link flow $f_{a_{jj'}}$ for $j = 1, ..., n$ and $j' = 1', ..., n'$. Let $a_{j'kl_2}$ denote the l_2^{th} link from node $y_{j'}$ to node z_k with associated link flow $f_{a_{j'kl_2}}$ for $j' = 1', ..., n'$; $k = 1, ..., m$, and $l_2 = 1, 2$. **The notation l_1 in a_{ijl_1} is used to distinguish between the two parallel links connecting x_i and y_j, while the notation l_2 in $a_{j'kl_2}$ is used to distinguish between the two parallel links between $y_{j'}$ and z_k.** Let $a_{(n+1)D}$ denote the link connecting node y_{n+1} and node D with associated link flow $f_{a_{(n+1)D}}$. Finally, let a_{kD} denote the link joining node z_k to node D with associated link flow $f_{a_{kD}}$.

We group the link flows into vectors as follows: the link flows $\{f_{a_{i(n+1)}}\}$; $i = 1, \ldots, m$ into the vector f^0; the link flows $\{f_{a_{ijl_1}}\}$; $i = 1, \ldots, m$; $j = 1, \ldots, n$; $l_1 = 1, 2$ into the vector f^1; the link flows $\{f_{a_{ik}}\}$; $i = 1, \ldots, m$; $k = 1, \ldots, o$ into the vector f^2, and the link flows $\{f_{a_{jj'}}\}$; $j = 1, \ldots, n$; $j' = 1', \ldots, n'$ into the vector f^3. Finally, we group the the link flows $\{f_{a_{j'kl_2}}\}$ into the vector f^4, and the $\{f_{a_{kD}}\}$; $k = 1, \ldots, o$ into the vector f^5.

In this supernetwork, there are three "types" of paths joining an origin node x_i to destination node D. The first type of path consists of four links: a_{ijl_1}, $a_{jj'}$, $a_{j'kl_2}$, and a_{kD}; for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $j' = 1', \ldots, n'$; $l_1 = 1, 2$; $l_2 = 1, 2$, and $k = 1, \ldots, o$ with a typical such path denoted by $p_{ijl_1j'kl_2}$ and with the flow on the path denoted by $x_{p_{ijl_1j'kl_2}}$. The second type of path consists of two links: a_{ik} and a_{kD}; for $i = 1, \ldots, m$; $k = 1, \ldots, o$, with a typical such path denoted by p_{ik} and its flow denoted by $x_{p_{ik}}$. The third type also consists of two links: $a_{i(n+1)}$ and $a_{(n+1)D}$, where $i = 1, \ldots, m$ and is denoted by $p_{i(n+1)}$ with its flow denoted by $x_{p_{i(n+1)}}$. There are $mo + m + 4mn^2o$ paths in G_S. We let d_{w_i} denote the fixed demand associated with O/D pair w_i and λ_{w_i} denotes the travel disutility for w_i.

Note that the following conservation of flow equations (cf. (21)), with the links, paths, and assocuated flows as defined above, must hold on the supernetwork:

$$f_{a_{ijl_1}} = \sum_{j'=1'}^{n'} \sum_{k=1}^{o} \sum_{l_2=1}^{2} x_{p_{ijl_1j'kl_2}}, \quad i = 1, \ldots, m; j = 1, \ldots, n; l_1 = 1, 2, \tag{27}$$

$$f_{a_{jj'}} = \sum_{i=1}^{m} \sum_{l_1=1}^{2} \sum_{k=1}^{o} \sum_{l_2=1}^{2} x_{p_{ijl_1j'kl_2}}, \quad j = 1, \ldots, n; j' = 1', \ldots, n', \tag{28}$$

$$f_{a_{j'kl_2}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l_1=1}^{2} x_{p_{ijl_1j'kl_2}}, \quad j' = 1', \ldots, n'; k = 1, \ldots, o; l_2 = 1, 2, \tag{29}$$

$$f_{a_{kD}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{j'=1'}^{n'} \sum_{l_2=1}^{2} x_{p_{ijl_1j'kl_2}} + \sum_{i=1}^{m} x_{p_{ik}}, \quad k = 1, \ldots, o, \tag{30}$$

$$f_{a_{ik}} = x_{p_{ik}}, \quad i = 1, \ldots, m; k = 1, \ldots, o, \tag{31}$$

$$f_{a_{i(n+1)}} = x_{p_{i(n+1)}}, \quad i = 1, \ldots, m, \tag{32}$$

$$f_{a_{(n+1)D}} = \sum_{i=1}^{m} x_{p_{i(n+1)}}. \tag{33}$$

Also, we have that (cf. (20))

$$d_{w_i} = \sum_{j=1}^{n} \sum_{l_1=1}^{2} \sum_{j'=1'}^{n'} \sum_{k=1}^{o} \sum_{l_2=1}^{2} x_{p_{ijl_1j'kl_2}} + \sum_{k=1}^{o} x_{p_{ik}} + x_{p_{i(n+1)}}, \quad i = 1, \ldots, m. \tag{34}$$

If all the path flows are nonnegative and (27) – (34) are satisfied, then the feasible path flow pattern induces a feasible link flow pattern.

We now can construct a feasible link flow pattern for G_S corresponding to a feasible financial flow pattern in the financial network model, $(Q^1, Q^2, Q^3, g, d) \in \mathcal{K}^4$, in the following way:

$$q_{ijl} \equiv f_{a_{ijl_1}}, \quad i = 1, \ \ldots, m; \ j = 1, \ldots, n; \ l = 1, 2; \ l_1 = l, \tag{35}$$

$$q_{i(n+1)} \equiv f_{i(n+1)}, \quad i = 1, \ldots, m, \tag{36}$$

$$g_j \equiv f_{a_{jj'}}, \quad j = 1, \ldots, n; \ j' = 1', \ldots, n', \tag{37}$$

$$q_{jkl} \equiv f_{a_{j'kl_2}}, \quad j = 1, \ldots, n; j' = 1', \ldots, n'; \ k = 1, \ldots, o; \ l = 1, 2; \ l_2 = l, \tag{38}$$

$$q_{ik} \equiv f_{a_{ik}}, \quad i = 1, \ldots, m; \ k = 1, \ldots, o, \tag{39}$$

$$d_k \equiv f_{a_{kD}}, \quad k = 1, \ldots, o. \tag{40}$$

Note that if (Q^1, Q^2, Q^3, g, d) is feasible then the associated link flow pattern constructed as in (35) – (40) is also feasible and the corresponding path flow pattern (cf. (27) – (34)) inducing this flow pattern is also feasible.

We now assign costs on the links of the supernetwork G_S as follows: to each link a_{ijl_1} assign a cost $c_{a_{ijl_1}}$ given by:

$$c_{a_{ijl_1}} \equiv 2V^i_{z_{jl}} \cdot q_i + 2V^j_{z_{il}} \cdot q_j + \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}}, \quad i = 1, \ldots, m; j = 1, \ldots, n; l_1 = l = 1, 2; \tag{41}$$

to each link $a_{i(n+1)}$ assign a cost $c_{a_{i(n+1)}}$:

$$c_{a_{i(n+1)}} \equiv 0, \quad i = 1, \ldots, m; \tag{42}$$

to each link a_{ik} assign a cost $c_{a_{ik}}$:

$$c_{a_{ik}} \equiv 2V^i_{z_{2n+k}} \cdot q_i + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^2, Q^3), \quad i = 1, \ldots, m; k = 1, \ldots, o; \tag{43}$$

to each link $a_{j'kl_2}$ assign a cost $c_{a_{j'kl_2}}$:

$$c_{a_{j'kl_2}} \equiv 2V^j_{z_{kl}} \cdot q_j + \frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^2, Q^3), \quad j = 1, \ldots, n; k = 1, \ldots, o; l_2 = l = 1, 2; \tag{44}$$

and to the link $a_{jj'}$ assign a cost: $c_{a_{jj'}}$

$$c_{a_{jj'}} \equiv \frac{\partial c_j(g)}{\partial g_j}, \quad j = 1, \ldots, n; j' = 1', \ldots, n'. \tag{45}$$

In addition, to the link $a_{(n+1)D}$ assign a cost $c_{a_{(n+1)D}}$:

$$c_{a_{(n+1)D}} \equiv M. \tag{46}$$

Finally, for each link a_{kD} assign a cost:

$$c_{a_{kD}} \equiv M - \rho_{3k}(d), \quad k = 1, \ldots, o, \tag{47}$$

where M is a scalar and defined by:

$$M \equiv \max_{k=1,\ldots,o} \sup_{d \in \mathcal{D}} (\rho_{3k}(d)) \tag{48}$$

where $\mathcal{D} \equiv \{d | d \in R^o_+ \text{ and } d_k \leq \sum_{i=1}^m S^i, \quad \forall k = 1, \ldots, o\}$.

The scalar M is used here to ensure that the cost on link a_{kD} is nonnegative. It will not appear in the final variational inequality formulation as will be proven below.

We now determine the costs associated with different paths in the supernetwork following (22) and the link costs defined by (41) – (47). A user ("traveler") of path $p_{ijl_1j'kl_2}$, for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $j' = 1', \ldots, n'$; $k = 1, \ldots, o$; $l_1 = 1, 2$; $l_2 = 1, 2$, on the supernetwork G_S in Figure 2, incurs a path cost $C_{p_{ijl_1j'kl_2}}$ given by

$$C_{p_{ijl_1j'kl_2}} = 2V^i_{z_{jl_1}} \cdot q_i + 2V^j_{z_{il_1}} \cdot q_j + \frac{\partial c_{ijl_1}(q_{ijl_1})}{\partial q_{ijl_1}} + \frac{\partial \hat{c}_{ijl_1}(q_{ijl_1})}{\partial q_{ijl_1}} + \frac{\partial c_j(g_j)}{\partial g_j}$$

$$+2V^j_{z_{kl_2}} \cdot q_j + \frac{\partial c_{jkl_2}(q_{jkl_2})}{\partial q_{jkl_2}} + \hat{c}_{jkl_2}(Q^2, Q^3) + M - \rho_{3k}(d). \tag{49}$$

A user of path p_{ik}, for $i = 1, \ldots, m$; $k = 1, \ldots, o$; on G_S in incurs a path cost $C_{p_{ik}}$ given by

$$C_{p_{ik}} = 2V^i_{z_{2n+k}} \cdot q_i + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^2, Q^3) + M - \rho_{3k}(d). \tag{50}$$

A user of path $p_{i(n+1)}$, for $i = 1, \ldots, m$, on G_S incurs a path cost $C_{p_{i(n+1)}}$ given by

$$C_{p_{i(n+1)}} = M. \tag{51}$$

Also, we assign the fixed travel demands associated with the O/D pairs as follows:

$$d_{w_i} = S^i, \quad i = 1, \ldots, m. \tag{52}$$

Consequently, the equilibrium conditions (cf. (24)) for the transportation network equilibrium model on the network G_S state that for every O/D pair w_i; $i = 1, \ldots, m$ and every path connecting the O/D pair w_i:

$$C_{p_{ijl_1j'kl_2}} \begin{cases} = \lambda_{w_i}, & \text{if} \quad x^*_{p_{ijl_1j'kl_2}} > 0 \\ \geq \lambda_{w_i}, & \text{if} \quad x^*_{p_{ijl_1j'kl_2}} = 0, \end{cases} \tag{53}$$

$$C_{p_{ik}} \begin{cases} = \lambda_{w_i}, & \text{if} \quad x^*_{p_{ik}} > 0 \\ \geq \lambda_{w_i}, & \text{if} \quad x^*_{p_{ik}} = 0, \end{cases} \tag{54}$$

$$C_{p_{i(n+1)}} \begin{cases} = \lambda_{w_i}, & \text{if} \quad x^*_{p_{i(n+1)}} > 0 \\ \geq \lambda_{w_i}, & \text{if} \quad x^*_{p_{i(n+1)}} = 0. \end{cases} \tag{55}$$

In view of (24) and (25), we may immediately write the variational inequality formulation in path flows whose solution satisfies (53) – (55). Indeed, we have that a path flow pattern x^* such that $x^* \geq 0$ and satisfies (34) is a transportation network equilibrium if and only if it satisfies the variational inequality problem:

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{l_1=1}^2 \sum_{j'=1'}^{n'} \sum_{k=1}^o \sum_{l_2=1}^2 C_{p_{ijl_1j'kl_2}} \times \left[x_{p_{ijl_1j'kl_2}} - x^*_{p_{ijl_1j'kl_2}}\right] + \sum_{i=1}^m \sum_{k=1}^o C_{p_{ik}} \times \left[x_{p_{ik}} - x^*_{p_{ik}}\right]$$

$$+ \sum_{i=1}^m C_{p_{i(n+1)}} \times \left[x_{p_{i(n+1)}} - x^*_{p_{i(n+1)}}\right] \geq 0, \quad \forall x \geq 0, \quad \text{and satisfying (34).} \tag{56}$$

Existence of a solution to variational inequality (56) follows immediately from the standard theory of variational inequalities (see, e.g., Dafermos (1980), Nagurney (1993), and Patriksson (1994)) since the path cost functions are continuous and the feasible set is compact (since the demands are equal to the sources of funds according to (52) which are finite).

We now show that the variational inequality formulation of the equilibrium conditions (53) – (55) in link form (cf. (26)) is equivalent to the variational inequalities (11) and (18b) governing the financial network problem. For the transportation network equilibrium problem on the supernetwork G_s, we know that, according to Theorem 2, a feasible link flow pattern is an equilibrium according to (24), equivalently, to (53) – (55), if and only if it satisfies:

$$\sum_{i=1}^{m} c_{a_{i(n+1)}}(f^{0*}) \times (f_{a_{i(n+1)}} - f^*_{a_{i(n+1)}}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l_1=1}^{2} c_{a_{ijl_1}}(f^{1*}, f^{2*}, f^{4*}) \times (f_{a_{ijl_1}} - f^*_{a_{ijl_1}})$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o} c_{a_{ik}}(f^{1*}, f^{2*}, f^{4*}) \times (f_{a_{ik}} - f^*_{a_{ik}}) + \sum_{j=1}^{n}\sum_{j'=1'}^{n'} c_{a_{jj'}}(f^{3*}) \times (f_{a_{jj'}} - f^*_{a_{jj'}})$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l_2=1}^{2} c_{a_{j'kl_2}}(f^{1*}, f^{2*}, f^{4*}) \times (f_{a_{j'kl_2}} - f^*_{a_{j'kl_2}}) + \sum_{k=1}^{o} c_{a_{kD}}(f^{5*}) \times (f_{a_{kD}} - f^*_{a_{kD}})$$

$$+c_{a_{(n+1)D}} \times (f_{a_{(n+1)D}} - f^*_{a_{(n+1)D}}) \geq 0, \quad \forall (f^0, f^1, f^2, f^3, f^4, f^5, f_{a_{(n+1)D}}) \in \mathcal{K}^6, \tag{57}$$

where

$$\mathcal{K}^6 \equiv \{(f^0, f^1, f^2, f^3, f^4, f^5, f_{a_{(n+1)D}}) | (f^0, f^1, f^2, f^3, f^4, f^5, f_{a_{(n+1)D}}) \in R_+^{2mn+mo+2no+m+o+1}$$

and there exist a nonnegative path flow vector x such that (27) – (34) and (52) hold}.

After substituting (35) – (40) and (41) – (47) into (57), we have that:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q^*_i + 2V^j_{z_{il}} \cdot q^*_j + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q^*_i + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q^*_{ik}]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}} \cdot q^*_j + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*})\right] \times \left[q_{jkl} - q^*_{jkl}\right] + \sum_{j=1}^{n} \frac{\partial c_j(g^*)}{\partial g_j} \times \left[g_j - g^*_j\right]$$

$$+\sum_{k=1}^{o}[M-\rho_{3k}(d^*)]\times[d_k-d_k^*]+M\times\left[\sum_{i=1}^{m}q_{i(n+1)}-\sum_{i=1}^{m}q_{i(n+1)}^*\right]\geq 0,$$

$$\forall(Q^1,Q^2,Q^3,g,d)\in\mathcal{K}^4. \tag{58}$$

After algebraic simplification of (58), we obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{jl}}^{i}\cdot q_i^*+2V_{z_{il}}^{j}\cdot q_j^*+\frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}\right]\times\left[q_{ijl}-q_{ijl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V_{z_{2n+k}}\cdot q_i^*+\frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}}+\hat{c}_{ik}(Q^{2*},Q^{3*})\right]\times[q_{ik}-q_{ik}^*]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V_{z_{kl}}\cdot q_j^*+\frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}}+\hat{c}_{jkl}(Q^{2*},Q^{3*})\right]\times\left[q_{jkl}-q_{jkl}^*\right]+\sum_{j=1}^{n}\frac{\partial c_j(g^*)}{\partial g_j}\times\left[g_j-g_j^*\right]$$

$$-\sum_{k=1}^{o}\rho_{3k}(d^*)\times[d_k-d_k^*]+M\times\left[\sum_{k=1}^{o}d_k+\sum_{i=1}^{m}q_{i(n+1)}-(\sum_{k=1}^{o}d_k^*+\sum_{i=1}^{m}q_{i(n+1)}^*)\right]\geq 0,$$

$$\forall(Q^1,Q^2,Q^3,g,d)\in\mathcal{K}^4. \tag{59}$$

Notice now that $\sum_{k=1}^{o}d_k+\sum_{i=1}^{m}q_{i(n+1)}=\sum_{i=1}^{m}S^i=\sum_{k=1}^{o}d_k^*+\sum_{i=1}^{m}q_{i(n+1)}^*$. Therefore, the last term in (59) is always equal to zero.

Hence, we can cancel out the last term in (59), which yields

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{jl}}^{i}\cdot q_i^*+2V_{z_{il}}^{j}\cdot q_j^*+\frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}\right]\times\left[q_{ijl}-q_{ijl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V_{z_{2n+k}}^{i}\cdot q_i^*+\frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}}+\hat{c}_{ik}(Q^{2*},Q^{3*})\right]\times[q_{ik}-q_{ik}^*]$$

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V_{z_{kl}}^{j}\cdot q_j^*+\frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}}+\hat{c}_{jkl}(Q^{2*},Q^{3*})\right]\times\left[q_{jkl}-q_{jkl}^*\right]+\sum_{j=1}^{n}\frac{\partial c_j(g^*)}{\partial g_j}\times\left[g_j-g_j^*\right]$$

$$-\sum_{k=1}^{o}\rho_{3k}(d^*)\times[d_k-d_k^*]\geq 0,\quad\forall(Q^1,Q^2,Q^3,g,d)\in\mathcal{K}^4. \tag{60}$$

Variational inequality (60) is precisely variational inequality (18b) governing the financial network problem with intermediation, which according to Corollary 2 is equivalent to variational inequality (11).

Hence, we have the following result:

Theorem 3

A solution $(Q^{1^*}, Q^{2^*}, Q^{3^*}, d^*) \in \mathcal{K}^2$ of the variational inequality (18b) governing a financial network equilibrium with intermediation and electronic transactions coincides with the feasible link flow pattern for the supernetwork G_S constructed above and satisfies variational inequality (60). Hence, it is a transportation network equilibrium pattern on the supernetwork G_S.

We now further elaborate about the new interpretation of the equilibrium conditions (53) – (55) in the context of the financial network with intermediation. Indeed, we now have a novel concept of equilibrium in terms of paths and path flows for the financial network which follows Wardrop's first principle of travel/traffic behavior (see also (24)) with the interpretation being that only those paths are used from a source agent to produce a financial product (with a financial product here being in the more general sense that non-investment also corresponds to a financial product) that are minimal in a cost sense. Hence, there is an underlying efficiency principle which yields the optimal paths for the production/transformation of financial flows from origins to the destination.

It is worth comparing the supernetwork equivalence established in this paper to the one constructed for decentralized supply chain networks by Nagurney (2005). In the case of multitiered supply chains, the supernetwork consists of a single origin node and as many destination nodes as there are demand markets for the product. In the supernetwork representing the financial network with intermediation and electronic transactions, in contrast, there is a single destination node and as many origin nodes as there are source agents with sources of financial funds. Also, in the case of supply chains, the supernetwork corresponds to an elastic demand transportation network equilibrium model developed by Dafermos and Nagurney (1984) whereas in the financial network context, the supernetwork corresponds to a fixed demand transportation network equilibrium model due to Smith (1979) and Dafermos (1980). Moreover, in the context of supply chains, there was no need to capture non-investment and the associated possible "slack" flows on links/paths.

We now provide an existence result using the supernetwork representation of the financial network model with intermediation, which we also emphasize is new in that, unlike the model in Nagurney and Ke (2003), here we consider demand price (or inverse demand) functions at the demand markets (rather than demand functions). Of course, if the demand functions are invertible then either model may be used. Such flexibility has been found to be very useful in transportation network applications as well as in the closely related spatial price equilibrium problems (see, e.g., Dafermos and Nagurney (1984) and Nagurney (1993) and the references therein).

In particular, we have

Theorem 4

There exists a solution to variational inequality (57) and, hence, to variational inequality (18b).

Proof: Follows from the standard theory of variational inequalities (see Dafermos (1980) and Nagurney (1993)) since the cost functions on the links are continuous and the feasible set is compact.

Note that Theorem 4 illustrates the potential power of the theoretical equivalence established here between financial network equilibrium with intermediation and transportation network equilibrium with fixed demands. Indeed, as we have shown, existence of solutions to the former problem permits for a direct and simple proof of existence of a solution to the latter.

5. Numerical Examples

In order to further demonstrate the potential applicability of the equivalence established in Section 4, but now, in terms of computations, we, in this Section, present six numerical financial network examples. We identify the supernetwork equivalence, and then determine the equilibrium on the supernetwork. The solution is then translated back to the financial network notation.

The examples were all solved, as noted in the Introduction, using the Euler method, which was proposed for the computation of solutions to fixed demand transportation network equilibrium problems by Nagurney and Zhang (1997); see also Nagurney and Zhang (1996). The algorithm is a special case of the general iterative scheme of Dupuis and Nagurney (1993) developed for the computation of solutions to variational inequality problems; equivalently, for the computation of stationary points of projected dynamical systems. Here we present the Euler method directly for the solution of the problems of concern, that is, transportation network equilibria. Note that the general iterative scheme of Dupuis and Nagurney (1993) also induces other algorithms, including Heun-type methods. Due to the equivalence of variational inequalities (25) and (26) and also since the theoretical results in Section 4 provide a new interpretation of financial network equilibria in terms of paths and path flows, we choose this algorithm which operates in the space of path flows.

Specifically, the Euler method applied to solve variational inequality (25) takes the form:

The Euler Method for the Computation of Fixed Demand Transportation Network Equilibria

At iteration τ determine

$$x^{\tau+1} = P_{\mathcal{K}^5}(x^\tau - a_\tau C(x^\tau)), \tag{61}$$

where C is the vector of path costs, $P_{\mathcal{K}^5}$ denotes the projection on the feasible set $\mathcal{K}^5$, and $\{a_\tau\}$ is a sequence of positive numbers satisfying $\sum_{\tau=1}^{\infty} a_\tau = \infty, \quad a_\tau \to 0 \text{ as } \tau \to \infty$, which is required for convergence (see Dupuis and Nagurney (1993), Nagurney and Zhang (1997)). Of course, (61) may be interpreted as a projection-type method in which the parameter a_τ varies from iteration to iteration (see also, Nagurney (1993)).

Expression (61) is equivalent to solving the quadratic programming problem: determine the vector of path flows $x^{\tau+1}$ according to:

$$x^{\tau+1} = \min_{x \in \mathcal{K}^5} \frac{1}{2} x^T \cdot x - (x^\tau - a_\tau C(x^\tau))^T \cdot x. \tag{62}$$

In view of the feasible set $\mathcal{K}^5$, problem (62), in turn, can be decomposed into as many subproblems as there are O/D pairs in the transportation/supernetwork, each of which is a quadratic programming problem with special structure that can be solved exactly and in closed form using exact equilibration, which was proposed by Dafermos and Sparrow (1969) and noted also by Nagurney and Zhang (1996). In particular, problem (62) is equivalent to the solution of: for each O/D pair w, compute:

$$\min \frac{1}{2} \sum_{p \in P_w} x_p^2 + \sum_{p \in P_w} h_p^\tau x_p \tag{63}$$

subject to:

$$\sum_{p \in P_w} x_p = d_w \tag{64}$$

$$x_p \geq 0, \quad \forall p \in P_w, \tag{65}$$

where

$$h_p^\tau = a_\tau C_p(x^\tau) - x_p^\tau. \tag{66}$$

Observe that this subproblem for each O/D pair w is over a network of special structure, in view of constraints (64) and (65). In particular, the specially structure network has disjoint paths, that is, the paths have no links in common. This is a notable feature of the Euler method in path flow variables. Convergence results can be found in Dupuis and Nagurney (1993) and in Nagurney and Zhang (1996, 1997).

For completeness and easy reference, we now state the exact equilibration algorithm where the iteration counter τ is suppressed, for simplicity. It can be used to solve (63) – (66) and can, hence, be embedded in the Euler method above to compute network equilibrium solutions. We will use this combination of algorithms for all the numerical examples in this Section.

Exact Equilibration Algorithm for O/D pair w

Step 0: Sort

Sort the fixed cost terms, h_p, $p \in P_w$, in nondescending order, and relabel accordingly. Assume, from this point on, that they are relabeled. Set $h_{p_{n_w}+1} = \infty$ where n_w denotes the number of paths in O/D pair w. Set $p = 1$.

Step 1: Computation

Compute:

$$\lambda_w^p = \frac{\sum_{i=1}^p h_p + d_w}{\sum_{i=1}^p 1}. \tag{67}$$

Step 2: Evaluation

If $h_p < \lambda_w^p \leq h_{p+1}$, then stop; set $q = p$, and go to Step 3. Otherwise, let $p := p + 1$, and go to Step 1.

Step 3: Update

Set

$$x_p = \lambda_w^q - h_p, \quad p = 1, \ldots, q, \tag{68}$$

$$x_p = 0, \quad p = q + 1, \ldots, n_w. \tag{69}$$

The Euler method had been coded in FORTRAN by the second author and had been applied to a spectrum of transportation network equilibrium problems in Nagurney and Zhang (1996, 1997). We used that code for the computation of solutions to all the numerical examples below. The computer system used was a Unix system at the University of Massachusetts at Amherst. For all the examples, the sequence $\{a_\tau\}$ was set to: $.1\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$. The convergence criterion was that the path flows in two subsequent iterations differed by no more than $\epsilon = .00001$. The Euler method was initialized by allocating the demand for each O/D pair equally among all the paths connecting that O/D pair.

The examples consisted of two source agents, two financial intermediaries, and two demand markets. These examples have the financial network structure depicted in Figure 3.

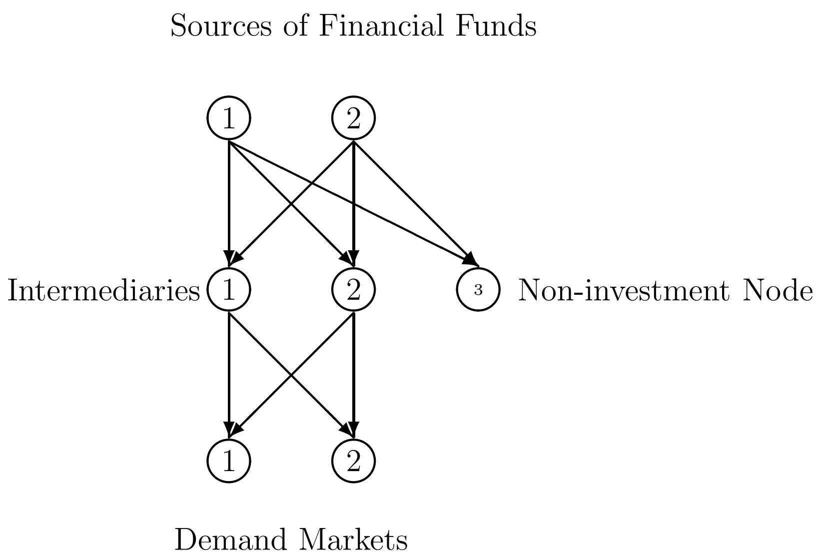

Figure 3: The Financial Network Structure of the Numerical Examples

For simplicity, we excluded electronic transactions.

Example 1

The financial holdings for the two source agents in the first example were: $S^1 = 10$ and $S^2 = 10$. The variance-covariance matrices V^i were identity matrices for all the source agents $i = 1, 2$. The variance-covariance matrices V^j, in turn, for intermediaries $j = 1, 2$ consisted of an identity submatrix associated with the q_{jk}; $k = 1, 2$ variables, with all other terms being equal to zero. We have suppressed the subscript l associated with the transaction cost functions since we have assumed a single (physical) mode of transaction only being available. Please refer to Table 1 for a compact exposition of the notation.

The transaction cost functions of the source agents associated with their transactions with the intermediaries were given by:

$$c_{ij}(q_{ij}) = 2q_{ij}^2 + q_{ij} + 1, \quad \text{for } i = 1, 2; j = 1, 2$$

with the partial derivative with respect to q_{ij} given by:

$$\frac{\partial c_{ij}}{\partial q_{ij}} = 4q_{ij} + 1, \quad \text{for } i = 1, 2; j = 1, 2.$$

The transaction cost functions of the intermediaries associated with transacting with the sources agents were given by:

$$\hat{c}_{ij}(q_{ij}) = 3q_{ij}^2 + 2q_{ij} + 1, \quad \text{for } i = 1, 2; j = 1, 2,$$

with the partial derivative with respect to q_{ij}:

$$\frac{\partial \hat{c}_{ij}}{\partial q_{ij}} = 6q_{ij} + 2, \quad \text{for } i = 1, 2; j = 1, 2.$$

The handling costs of the intermediaries were:

$$c_1(Q^1) = 0.5(q_{11} + q_{21})^2, \quad c_2(Q^1) = 0.5(q_{12} + q_{22})^2,$$

which are equivalent, respectively, to:

$$c_1(g) = 0.5g_1^2, \quad c_2(g) = 0.5g_2^2,$$

with the partial derivatives with respect to g_j; $j = 1, 2$:

$$\frac{\partial c_1}{\partial g_1} = g_1, \quad \frac{\partial c_2}{\partial g_2} = g_2,$$

where $g_j \equiv \sum_{i=1}^{2} q_{ij}, \quad \text{for } j = 1, 2.$

We assumed that in the transactions between the intermediaries and the demand markets, the transaction costs perceived by the intermediaries were all equal to zero, that is,

$$c_{jk} = 0, \quad \text{for } j = 1, 2; k = 1, 2.$$

The transaction costs between the intermediaries and the consumers at the demand markets, in turn, were given by:

$$\hat{c}_{jk} = q_{jk} + 2, \quad \text{for } j = 1, 2; k = 1, 2.$$

The inverse demand (demand market price) functions at the demand markets were:

$$\rho_{3k}(d) = -2d_k + 100, \quad \text{for } k = 1, 2. \tag{70}$$

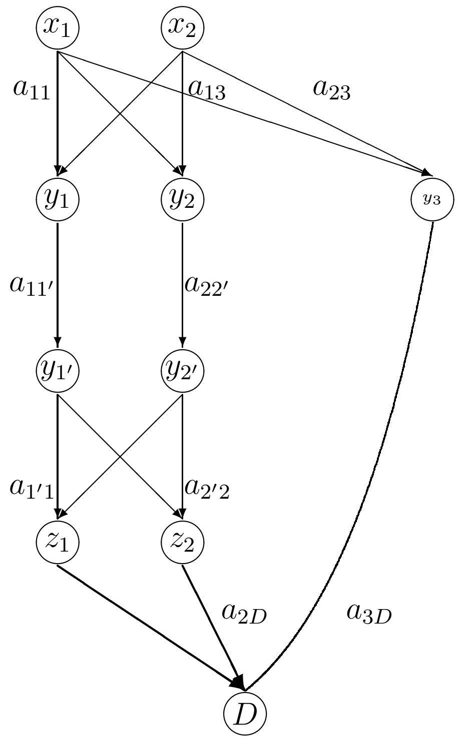

Figure 4: The Supernetwork G_S for the Numerical Examples

We now provide the supernetwork G_s for the reformulation of this this financial network as a fixed demand tranportation network equilibrium problem. The supernetwork is given in Figure 4 and consists of ten nodes, fifteen links, and two O/D pairs; $w_1 = (x_1, D)$ and $w_2 = (x_2, D)$.

Since we only considered physical transactions, as noted above, we can suppress the indices l_1 and l_2. Using now (27) – (40), the demands and the link flows in the supernetwork are as follows:

$$d_{w_i} = S_i = 10, \quad i = 1, 2, \tag{71}$$

$$f_{a_{ij}} = q_{ij}, \quad \text{for } i = 1, 2; j = 1, 2, \tag{72}$$

$$f_{a_{jj'}} = g_j, \quad \text{for } j = 1, 2; j' = 1', 2', \tag{73}$$

$$f_{a_{j'k}} = q_{jk}, \quad \text{for } j' = 1, 2; j' = 1', 2'; k = 1, 2, \tag{74}$$

$$f_{a_{kD}} = d_k, \quad \text{for } k = 1, 2. \tag{75}$$

The link costs (cf. (41) – (48)), hence, were defined as follows:

$$c_{a_{ij}} \equiv 2V^i_{z_j} \cdot q_i + 2V^j_{z_i} \cdot q_j + \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}} + \frac{\partial \hat{c}_{ij}(q_{ij})}{\partial q_{ij}}$$

$$= 2f_{a_{ij}} + 2f_{a_{ij}} + 4f_{a_{ij}} + 1 + 6f_{a_{ij}} + 2 = 14f_{a_{ij}} + 3, \quad \text{for } i = 1, 2; j = 1, 2,$$

$$c_{a_{i3}} \equiv 0, \quad i = 1, 2,$$

$$c_{a_{jj'}} \equiv \frac{\partial c_j(g)}{\partial g_j} = f_{a_{jj'}}, \quad \text{for } j = 1, 2; j' = 1', 2',$$

$$c_{a_{j'k}} \equiv 2V^j_{z_k} \cdot q_j + \frac{\partial c_{jk}(q_{jk})}{\partial q_{jk}} + \hat{c}_{jk}(Q^2, Q^3) = 2f_{a_{j'k}} + 0 + f_{a_{j'k}} + 2 = 3f_{a_{j'k}} + 2, \quad \text{for } j' = 1', 2'; k = 1, 2.$$

In this and in the next two examples, $M \equiv 100$, and, hence,

$$c_{a_{kD}} \equiv M - \rho_k(d) = 2f_{a_{kD}}, \quad \text{for } k = 1, 2,$$

$$c_{a_{3D}} \equiv M = 100.$$

We now list the paths for each O/D pair (refer to Figure 4). Since only physical transactions are involved, we suppress the subscripts l_1 and l_2, for simplicity, and we enumerate the paths as p_1, p_2, and so on, in which we list the specific links as in Figure 4. The paths in O/D pair w_1, denoted by the set P_{w_1} are, hence, given by:

$$p_1 = (a_{11}, a_{11'}, a_{1'1}, a_{1D}), \quad p_2 = (a_{11}, a_{11'}, a_{1'2}, a_{2D}),$$

$$p_3 = (a_{12}, a_{22'}, a_{2'1}, a_{1D}), \quad p_4 = (a_{12}, a_{22'}, a_{2'2}, a_{2D}),$$

$$p_5 = (a_{13}, a_{3D}).$$

The paths in O/D pair w_2 denoted by the set P_{w_2} are given by:

$$p_6 = (a_{21}, a_{11'}, a_{1'1}, a_{1D}), \quad p_7 = (a_{21}, a_{11'}, a_{1'2}, a_{2D}),$$

$$p_8 = (a_{22}, a_{22'}, a_{2'1}, a_{1D}), \quad p_9 = (a_{22}, a_{22'}, a_{2'2}, a_{2D}),$$

$$p_{10} = (a_{23}, a_{3D}).$$

The Euler method converged in 24 iterations and yielded the equilibrium path flows and link flows given in Table 2 and Table 3, respectively. In this example, all paths were used in equilibrium for each O/D pair and the path costs were all equal to 100.

The computed equilibrium link flows can be converted to the financial flows using expressions (72) – (75). In addition, the demand market prices can be obtained from (70), which yields: $\rho_{31} = \rho_{32} = 83.4$. Similar results can be obtained for the subsequent examples. Note that in this Example, each source agent did not invest the amount 1.74.

As further verification of the theory in this paper, we then converted the demand market price functions into demand functions by inverting them, which yielded the demand functions:

$$d_1 = -.5\rho_{31} + 50, \quad d_2 = -.5\rho_{32} + 50.$$

The resulting numerical financial network model with intermediation with the data as in Example 1 but with the demand functions would correspond to the model developed in Nagurney and Ke (2003) (but without electronic transactions). We then solved this version of the model using the Euler method but applied directly to the financial network model with demand functions as described in Nagurney and Ke (2003) and obtained, as expected, the identical equilibrium financial flow pattern, that is:

$$Q^{1*} := q_{ij}^* = 4.13, \text{ for } i = 1, 2; j = 1, 2;$$

$$Q^{3*} := q_{jk}^* = 4.13, \text{ for } j = 1, 2; k = 1, 2.$$

In addition, the algorithm proposed in Nagurney and Ke (2003) computed the demand market prices: $\rho_{31}^* = \rho_{32}^* = 83.4$.

Example 2

In the second numerical example, the data were identical to those in Example 1, except that the financial holdings of the source agents S_i; $i = 1, 2$, were now both equal to 6. Hence, the source agents had fewer financial holdings to allocate in this example as compared to Example 1.

The Euler method converged in 2 iterations. The computed equilibrium path flow and link flow patterns are given, respectively, in Table 2 and Table 3.

In this example, for both O/D pairs, in equilibrium, the costs of the used paths were all equal to 74. In the case of the paths containing the non-investment links, with path costs equal 100, the flows were zero; in other words, all the financial funds are invested. Hence, as in the conditions (53) – (55), in equilibrium, only those paths connecting an O/D pair with minimal costs are used, that is, have positive flow on them.

The equilibrium financial flow pattern can be obtained directly from expressions: (72) – (75).

We then also solved the financial network model with intermediation with the data for Example 2, but with the demand functions (rather than their inverses) using, again using the algorithm proposed in Nagurney and Ke (2003). We obtained, as expected,

$$Q^{1*} := q^*_{ij} = 3.00, \text{ for } i = 1, 2; j = 1, 2;$$

$$Q^{3*} := q^*_{jk} = 3.00, \text{ for } j = 1, 2; k = 1, 2.$$

The demand market equilibrium prices obtained were: $\rho^*_{31} = \rho^*_{32} = 88$, which are exactly the values obtained by substitution of the $d^*_1 = d^*_2 = 6$. These results coincide with the corresponding equilibrium link flow pattern obtained by the Euler method applied to the supernetwork as above.

Example 3

The third numerical example had the same data as Example 1 except that the handling cost of the **second intermediary**, $c_2(Q^1)$, was given by:

$$c_2(Q^1) = (q_{12} + q_{22})^2,$$

or, equivalently,

$$c_2(g) = g_2^2$$

with the marginal handling cost:

$$\frac{\partial c_2(g)}{\partial g_2} = 2g_2,$$

so that on the supernetwork we had now that

$$c_{a_{22'}} = 2f_{a_{22'}}.$$

The Euler method converged in 13 iterations. The equilibrium path flows and link flows are reported in Tables 2 and 3. In this example, for both O/D pairs, all paths were used, and the path costs were all equal to 100.

We then also solved the financial network model with intermediation with the data for Example 3, but with the demand functions (rather than their inverses) via the algorithm proposed by Nagurney and Ke (2003) and obtained the identical results for the financial flow pattern corresponding to the link flows. The equilibrium demand market prices were now $\rho_{31}^* = \rho_{32}^* = 84.05$.

Example 4

Example 4 was constructed from Example 1 and had the identical data except that the demand market price (inverse demand) functions were now changed to:

$$\rho_{31}(d) = -1.14d_1 + .858d_2 + 281.71, \quad \rho_{32}(d) = -1.14d_2 + .858d_1 + 281.71,$$

so that M was now equal to 281.71. Hence, all the link cost functions on the supernetwork remained the same except that now we had that:

$$c_{a_{1D}} = 1.14f_{a_{1D}} - .858f_{a_{2D}}, \quad c_{a_{2D}} = 1.14f_{a_{2D}} - .858f_{a_{1D}}, \quad c_{3D} = 281.71.$$

The Euler method converged in 2 iterations. The computed equilibrium path flows and links flows are given in Tables 2 and 3. In this example, for both O/D pairs, all paths were used, except for the two paths (one per O/D pair) containing the non-investment links. The costs on the used paths were all equal to 102.82; the costs on the two unused paths were equal to 281.71.

Example 5

Example 5, in turn, had the same data as Example 2, but with the inverse demand functions as in Example 4, so that the costs on links: a_{1D}, a_{2D}, and a_{3D} were as in Example 4. The computed equilibrium link flow pattern was identical to that in Example 2 as was the equilibrium path flow pattern. However, the cost on all the used paths was now equal to 63.69. The cost on the two unused paths was equal to 281.71. The Euler method converged in 2 iterations. Refer to Tables 2 and 3 for details.

Example 6

Example 6 was constructed from Example 3 and it had the same data as Example 3 except that the demand market price functions were as in Examples 4 and 5.

The Euler method now converged in 8 iterations and yielded the equilibrium path flows and link flows reported, respectively, in Table 2 and in Table 3. The cost on the used paths (for both O/D pairs) was 107.57 and the cost on the unused paths (also for both O/D pairs) was 281.71.

We emphasize that the above framework enables numerous simulations in which the effects of changes to the transaction cost, handling cost, and inverse demand functions, as well as to the financial holdings, etc., can be made and the impacts on the resulting flows determined. Indeed, we have conducted additional numerical experiments in which the financial holdings were distinct for the source agents and the inverse demand functions were as well and we were able to obtain equilibrium solutions which fully supported the theoretical results in Section 4. Of course, one can also investigate the effects of the addition/deletion of source agents, and/or financial intermediaries, and/or demand markets.

Table 2: Path Flow Solutions to Examples 1, 2, 3, 4, 5, and 6

Path Flows	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6
$x^*_{p_1}$	2.07	1.50	2.08	2.50	1.50	2.63
$x^*_{p_2}$	2.07	1.50	2.08	2.50	1.50	2.63
$x^*_{p_3}$	2.07	1.50	1.88	2.50	1.50	2.38
$x^*_{p_4}$	2.07	1.50	1.88	2.50	1.50	2.38
$x^*_{p_5}$	1.74	0.00	2.07	0.00	0.00	0.00
$x^*_{p_6}$	2.07	1.50	2.08	2.50	1.50	2.63
$x^*_{p_7}$	2.07	1.50	2.08	2.50	1.50	2.63
$x^*_{p_8}$	2.07	1.50	1.88	2.50	1.50	2.38
$x^*_{p_9}$	2.07	1.50	1.88	2.50	1.50	2.38
$x^*_{p_{10}}$	1.74	0.00	2.07	0.00	0.00	0.00

Table 3: Link Flow Solutions to Examples 1, 2, 3, 4, 5, and 6

Link Flows	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6
$f^*_{a_{11}}$	4.13	3.00	4.16	5.00	3.00	5.25
$f^*_{a_{12}}$	4.13	3.00	3.77	5.00	3.00	4.75
$f^*_{a_{13}}$	1.74	0.00	2.07	0.00	0.00	0.00
$f^*_{a_{21}}$	4.13	3.00	4.16	5.00	3.00	5.25
$f^*_{a_{22}}$	4.13	3.00	3.77	5.00	3.00	4.75
$f^*_{a_{23}}$	1.74	0.00	2.07	0.00	0.00	0.00
$f^*_{a_{11'}}$	8.26	6.00	8.33	10.00	6.00	10.50
$f^*_{a_{22'}}$	8.26	6.00	7.54	10.00	6.00	9.50
$f^*_{a_{1'1}}$	4.13	3.00	4.16	5.00	3.00	5.25
$f^*_{a_{1'2}}$	4.13	3.00	4.16	5.00	3.00	5.25
$f^*_{a_{2'1}}$	4.13	3.00	3.77	5.00	3.00	4.75
$f^*_{a_{2'2}}$	4.13	3.00	3.77	5.00	3.00	4.75
$f^*_{a_{1D}}$	8.26	6.00	7.93	10.00	6.00	10.00
$f^*_{a_{2D}}$	8.26	6.00	7.93	10.00	6.00	10.00
$f^*_{a_{3D}}$	3.48	0.00	4.14	0.00	0.00	0.00

6. Summary and Conclusions

In this paper, we have considered two distinct classes of network problems, with accompanying research, corresponding to financial network problems with intermediation and transportation network equilibrium problems. Financial network problems with intermediation which can also include electronic financial networks are characterized by decision-makers associated with tiers of the financial network who allocate/transform financial resources from those with sources of financial funds, through financial intermediaries, into distinct financial products associated with the demand markets. Transactions may also take place directly between those with sources of funds and the consumers at the demand markets. In the case of transportation network equilibrium problems, travelers or users of the network seek to determine their cost-minimizing routes of travel, acting unilaterally, with the governing equilibrium concept being that of Wardrop (1952) in that only those paths connecting each origin/destination pair of nodes will be used such that their costs are equal and minimal.

In this paper, we first presented a new model of financial network equilibrium with intermediation and electronic transactions in which the inverse demand functions, that is, the demand market price functions were assumed known and given. The model was based on the financial network model of Nagurney and Ke (2003) in which the demand functions were assumed. We derived the governing equilibrium conditions and presented the variational inequality formulation. We then recalled the transportation network equilibrium model due to Smith (1979) and Dafermos (1980) in which the travel demands associated with the origin/destination pairs are fixed and given. We also presented the corresponding variational inequality formulations in both path flows and link flows for the transportation network equilibrium model.

Subsequently, we constructed the supernetwork representation of the financial network problem, which corresponds to an isomorphic transportation network equilibrium problem with fixed demands, and we proved that the respective variational inequalities coincide. This equivalence allowed us to provide a novel interpretation of the equilibrium conditions of the financial network problems with intermediation in terms of paths and path flows. We then exploited this equivalence theoretically by providing a direct existence proof of the solution to the new financial network model with intermediation.

Finally, since the topic of transportation network equilibrium modeling, analysis, and computations has been an active subject of research, beginning with the classical book of Beckmann, McGuire, and Winsten (1956), this connection provides us with many opportunities for the transferral of results in that field to financial networks with intermediation. We then proposed and applied an algorithm for the computation of solutions to fixed demand transportation network equilibrium problems by Nagurney and Zhang (1997) to compute solutions to six numerical financial network problems. The computational results yielded information that was not previously available since both the equilibrium path flows as well as the equilibrium link flows were now obtained.

The results in this paper are companions to the recent paper of Nagurney (2005) which established that decentralized, multitiered supply chain network equilibrium problems can be reformulated as transportation network equilibrium problems, but with elastic demands, and on a supernetwork entirely distinct from the one obtained here for financial network problems with intermediation.

Future research may include exploring the potential of the results obtained in this paper for reformulating international financial networks with intermediation originated by Nagurney and Cruz (2003a, b). In addition, it would be very interesting to explore the solution of large-scale financial network problems computationally using a spectrum of algorithms from the transportation science literature. Finally, it would be illuminating to derive stability and sensitivity analysis results for financial networks with intermediation as have been obtained for transportation networks (see, e.g., Nagurney (1993)). Of course, we note that here we have assumed that the risk is modeled as a variance and that the use of more advanced risk measures would be another interesting extension. In addition, in the future, we may explore other multiobjective techniques, rather than use an additive function to represent the multicriteria decision-making behavior, reflecting, in effect, risk minimization and profit or net revenue maximization, of both the sources of financial funds and the financial intermediaries in our financial model.

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