Dynamics of International Financial Networks with Risk Management

Anna Nagurney and Jose Cruz
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

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Abstract

In this paper, we develop an international financial network model in which the sources of funds and the intermediaries are multicriteria decision-makers and are concerned with both net revenue maximization and risk minimization. The model allows for both physical as well as electronic transactions and considers three tiers of decision-makers who may be located in distinct countries and may conduct their transactions in different currencies. We describe the behavior of the various decision-makers, along with their optimality conditions, and derive the variational inequality formulation of the governing equilibrium conditions. We then propose a dynamic adjustment process which yields the evolution of the financial flows and prices and demonstrate that it can be formulated as a projected dynamical system. We also provide qualitative properties including stability analysis results. Finally, we discuss a discrete-time algorithm which can be applied to track the dynamic trajectories and yields the equilibrium financial flows and prices. We illustrate both the modeling framework as well as the computational procedure with several numerical international financial network examples.

1. Introduction

The new global financial marketplace is characterized by a wide spectrum of choices available for financial transactions internationally plus a large number of decision-makers involved, be they sources of financial funds, intermediaries, and/or, ultimately, the consumers of the various financial products. In addition, new currencies have been introduced, such as the euro, along with new financial products in different currencies and countries. Moreover, the landscape of financial decision-making has been transformed through advances in telecommunications and, in particular, the Internet, providing business, brokerages, and consumers with new modes of transaction, new types of products and services, as well as new distribution channels.

Indeed, the growth of technology has allowed consumers and business to explore and conduct their financial transactions not only within the perimeters of national boundaries but outside them as well. At the same time that finance has become increasingly globalized, partially due to the increasing role of electronic finance, decision-makers are being faced with a growing importance of addressing risk due to heightened uncertainties. Such complex realities raise major challenges and opportunities for financial quantitative modeling, analysis, and computation.

In this paper, we consider *financial networks* and we develop an international financial network model which includes intermediaries and also allows for the decision-makers to transact either physically or electronically. The framework that we propose predicts the equilibrium financial flows between tiers of the international financial network as well as the equilibrium prices associated with the different tiers. In addition, it describes the evolution of the financial flows and prices over time. The methodologies utilized in this paper include finite-dimensional variational inequality theory (cf. Nagurney (1999) and the references therein) as well as projected dynamical systems theory (Nagurney and Zhang (1996a) and Nagurney and Siokos (1997)).

The model that we develop here, both in the static and dynamic versions, is sufficiently general to be able to handle as many countries, sources of funds, currencies, and financial products as mandated by the particular application. Although the topic of electronic finance, in general, has received much attention lately (cf. McAndrews and Stefanidis (2000), Claessens, Glaessner, and Klingebiel (2000, 2001), Fan, Staeelert, and Whinston (2000), Economides (2001), Allen, Hawkins, and Sato (2001), Sato and Hawkins (2001), Banks (2001), Claessens, et al. (2003), and the references therein), there has been very little work in the modeling, analysis, and solution of such problems, except for that of Nagurney and Ke (2003) who focused, however, on single country modeling and Nagurney and Cruz (2002, 2003), who considered only static formulations.

Indeed, recently, financial networks have been utilized to develop general models with multiple tiers of decision-makers, including intermediaries, by Nagurney and Ke (2001, 2003) and Nagurney and Cruz (2002, 2003). These models differ from the financial network models described in Nagurney and Siokos (1997) in that the behavior of the individual decision-makers associated with the distinct tiers of the network is explicitly captured and modeled. Moreover, unlike the framework considered in Thore (1980), more general, including nonlinear and asymmetric functions can be handled. However, the above-noted models of financial networks with intermediaries have focused exclusively on optimization/equilibrium formulations using finite-dimensional variational inequality formulations (see also Nagurney (1999)). Here, in contrast, we turn to extending the above frameworks to incorporate dynamics.

In particular, in this paper, we not only extend the international financial network models of Nagurney and Cruz (2002, 2003) to incorporate multicriteria decision-making explicitly for the purpose of risk management, but we also introduce dynamics. The theory utilized is the theory of *projected dynamical systems* (cf. Nagurney and Zhang (1996a) and the references therein) which allows for the formulation and qualitative analysis of the interaction of decision-makers whose decisions are subject to constraints. Projected dynamical systems have been used to-date to formulate the dynamics of a variety of problems including those arising in oligopolies, spatial price problems, traffic network problems, and financial networks (but without intermediation) (cf., respectively, Nagurney, Dupuis, and Zhang (1994), Nagurney and Zhang (1996b), Nagurney and Zhang (1997), and Dong, Zhang, and Nagurney (1996)). It is applied here for the first time in multitiered, multicriteria international financial network problems with intermediation (as well as with electronic transactions and risk management).

This paper is organized as follows. In Section 2, we develop the international financial network model with multicriteria decision-makers. We describe the various decision-makers and their behavior, and construct the equilibrium conditions, along with the variational inequality formulation.

In Section 3, we propose a dynamic adjustment process by which the financial flows and prices evolve and prove that the set of stationary points of the projected dynamical system coincides with the set of solutions of the variational inequality. We then, in Section 4, obtain qualitative properties of both the equilibrium pattern as well as the dynamic trajectories.

In Section 5, we present the algorithm, which is a discrete-time algorithm, and which provides a time discretization of the dynamic trajectories. The algorithm is then applied in Section 6 to compute the financial flows and prices in several international financial network examples. We conclude the paper with a summary and a discussion in Section 7.

2. The International Financial Network Model with Risk Management

In this Section, we develop the international financial network model with three tiers of decision-makers, which allows for both physical and electronic transactions. The multicriteria decision-makers are the sources of funds as well as the intermediaries both of whom are concerned with net revenue maximization and risk minimization. In this Section, we describe the static version of the model, which is then extended to its dynamic counterpart in Section 3. The set of equilibrium points of the former are, subsequently, shown to coincide with the set of stationary points of the latter.

The model consists of L countries, with a typical country denoted by l or $\hat{l}$; I "source" agents in each country with sources of funds, with a typical source agent denoted by i, and J financial intermediaries with a typical financial intermediary denoted by j. Examples of source agents are households and businesses, whereas examples of financial intermediaries include banks, insurance companies, investment companies, brokers, including electronic brokers, etc. Intermediaries in our framework need not be country-specific but, rather, may be virtual.

We assume that each source agent can transact directly electronically with the consumers through the Internet and can also conduct his financial transactions with the intermediaries either physically or electronically in different currencies. There are H currencies in the international economy, with a typical currency being denoted by h. Also, we assume that there are K financial products which can be in distinct currencies and in different countries with a typical financial product (and associated with a demand market) being denoted by k. Hence, the financial intermediaries in the model, in addition to transacting with the source agents, also determine how to allocate the incoming financial resources among distinct uses, which are represented by the demand markets with a demand market corresponding to, for example, the market for real estate loans, household loans, or business loans, etc., which, as mentioned, can be associated with a distinct country and a distinct currency combination. We let m refer to a mode of transaction with $m = 1$ denoting a physical transaction and $m = 2$ denoting an electronic transaction via the Internet.

The international financial network with electronic transactions is now described and depicted graphically in Figure 1. The top tier of nodes consists of the agents in the different

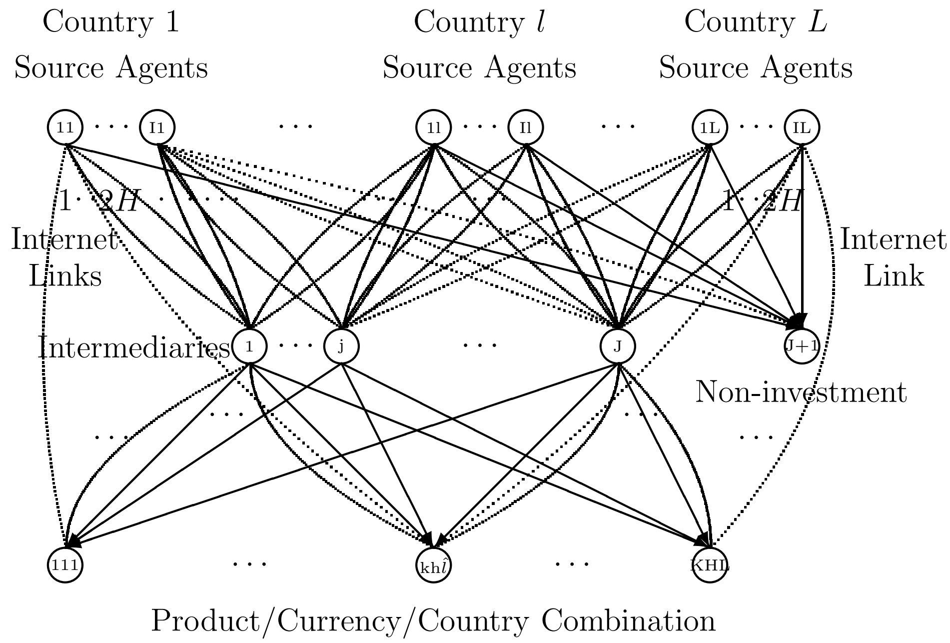


Figure 1: International financial network with intermediaries and with electronic transactions (non-investment allowed)

countries with sources of funds, with agent i in country l being referred to as agent il and associated with node il. There are, hence, IL top-tiered nodes in the network. The middle tier of nodes consists of the financial intermediaries (which need not be country-specific), with a typical intermediary j associated with node j in this (second) tier of nodes in the network. The bottom tier of nodes consists of the demand markets, with a typical demand market for product k in currency h and country $\hat{l}$ associated with node $kh\hat{l}$. There are, as depicted in Figure 1, J middle (or second) tiered nodes corresponding to the intermediaries and KHL bottom (or third) tiered nodes in the international financial network. In addition, we add a node $J+1$ to the middle tier of nodes in order to represent the possible non-investment (of a portion or all of the funds) by one or more of the source agents (see also Nagurney and Ke (2003)).

Now that we have identified the nodes in the international financial network we turn to the identification of the links joining the nodes in a given tier with those in the next tier.

We also associate the financial flows with the appropriate links. We assume that each agent i in country l has an amount of funds S^{il} available in the base currency. Since there are assumed to be H currencies and 2 modes of transaction (physical or electronic), there are $2H$ links joining each top tier node il with each middle tier node j; $j = 1, \ldots, J$, with the first H links representing physical transactions between a source and intermediary, and with the corresponding flow on such a link given, respectively, by x^{il}_{jh1}, and the subsequent H links representing electronic transactions with the corresponding flow given, respectively, by x^{il}_{jh2}. Hence, x^{il}_{jh1} denotes the nonnegative amount invested (across all financial instruments) by source agent i in country l in currency h transacted through intermediary j using the physical mode whereas x^{il}_{jh2} denotes the analogue but for an electronic transaction. We group the financial flows for all source agents/intermediaries/modes into the column vector $x^1 \in R^{2ILJH}_+$. In addition, a source agent i in country l may transact directly with the consumers at demand market k in currency h and country $\hat{l}$ via an Internet link. The nonnegative flow on such a link joining node il with node $kh\hat{l}$ is denoted by $x^{il}_{kh\hat{l}}$. We group all such financial flows, in turn, into the column vector $x^2 \in R^{ILKHL}_+$. Also, we let x^{il} denote the $(2JH + KHL)$-dimensional column vector associated with source agent il with components:$\{x^{il}_{jhm}, x^{il}_{kh\hat{l}}; j = 1, \ldots, J; h = 1, \ldots, H; m = 1, 2; k = 1, \ldots, K; \hat{l} = 1, \ldots, L\}$. Furthermore, we construct a link from each top tiered node to the second tiered node $J+1$ and associate a flow s^{il} on such a link emanating from node il to represent the possible nonnegative amount not invested by agent i in country l.

Each intermediary node j; $j = 1, \ldots, J$, may transact with a demand market via a physical link, and/or electronically via an Internet link. Hence, from each intermediary node j, we construct two links to each node $kh\hat{l}$, with the first such link denoting a physical transaction and the second such link – an electronic transaction. The corresponding flow, in turn, which is nonnegative, is denoted by $y^j_{kh\hat{l}m}$; $m = 1, 2$, and corresponds to the amount of the financial product k in currency h and country $\hat{l}$ transacted from intermediary j via mode m. We group the financial flows between node j and the bottom tier nodes into the column vector $y^j \in R^{2KHL}_+$. All such financial flows for all the intermediaries are then further grouped into the column vector $y \in R^{2JKHL}_+$.

The notation for the prices is now given. There will be prices associated with each of the tiers of nodes in the international financial network. The prices are assumed to be in the base currency. Let ρ^{il}_{1jhm} denote a price (in the base currency) associated with the financial instrument in currency h transacted via mode m as quoted by intermediary j to agent il and group the top tier prices into the column vector $\rho_1 \in R^{2ILJH}_+$. In addition, let $\rho^{il}_{1kh\hat{l}}$ denote a price, also in the base currency, associated with the financial instrument as quoted by demand market $kh\hat{l}$ to agent il and group these top tier prices into the column vector $\rho_{12} \in R^{ILKHL}_+$. Let $\rho^{j}_{2kh\hat{l}m}$, in turn, denote the price associated with intermediary j for product k in currency h and country $\hat{l}$ transacted via mode m and group all such prices into the column vector $\rho_2 \in R^{2JKHL}_+$. Also, let $\rho_{3kh\hat{l}}$ denote the price of the financial product k in currency h and in country $\hat{l}$, and defined in the base currency, and group all such prices into the column vector $\rho_3 \in R^{KHL}_+$. Finally, let e_h denote the rate of appreciation of currency h against the base currency, which can be interpreted as the rate of return earned due to exchange rate fluctuations (see Nagurney and Siokos (1997)). These "exchange" rates are grouped into the column vector $e \in R^H_+$.

We now turn to describing the behavior of the various decision-makers represented by the three tiers of nodes in Figure 1. We first focus on the top-tier decision-makers. We then turn to the intermediaries and, subsequently, to the consumers at the demand markets.

The Behavior of the Agents with Sources of Funds and their Optimality Conditions

We denote the transaction cost associated with source agent il transacting with intermediary j in currency h via mode m by c^{il}_{jhm} (and measured in the base currency) and assume that:

$$c^{il}_{jhm} = c^{il}_{jhm}(x^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{1a}$$

that is, the cost associated with source agent i in country l transacting with intermediary j in currency h depends on the volume of the transaction. We denote the transaction cost associated with source agent il transacting with demand market k in country $\hat{l}$ in currency h via the Internet link by $c^{il}_{kh\hat{l}}$ (and also measured in the base currency) and assume that:

$$c^{il}_{kh\hat{l}} = c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}), \quad \forall i, l, k, h, \hat{l}, \tag{1b}$$

that is, the cost associated with source agent i in country l transacting with the consumers for financial product k in currency h and country $\hat{l}$. The transaction cost functions are assumed to be convex and continuously differentiable and depend on the volume of flow of the transaction.

The total transaction costs incurred by source agent il are equal to the sum of all of his transaction costs associated with dealing with the distinct intermediaries and demand markets in the different currencies. His revenue, in turn, is equal to the sum of the price (rate of return plus the rate of appreciation) that the agent can obtain times the total quantity obtained/purchased. Let now ρ^{il*}_{1jhm} denote the actual price charged agent il for the instrument in currency h by intermediary j by transacting via mode m and let $\rho^{il*}_{1kh\hat{l}}$, in turn, denote the actual price associated with source agent il transacting electronically with demand market $kh\hat{l}$. Similarly, let e^*_h denote the actual rate of appreciation in currency h. We later discuss how such prices are recovered.

We assume that each source agent seeks to maximize his net return with the net revenue maximization criterion for source agent il being given by:

$$\text{Maximize} \quad \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho^{il*}_{1jhm}+e^*_h)x^{il}_{jhm}+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho^{il*}_{1khl}+e^*_h)x^{il}_{kh\hat{l}}$$

$$-\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}c^{il}_{jhm}(x^{il}_{jhm})-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}). \tag{2}$$

Note that the first two terms in (2) reflect the revenue whereas the last two terms represent the costs.

In addition to the criterion of net revenue maximization, we also assume that each source of funds is concerned with risk minimization. Here, for the sake of generality, we assume, as given, a risk function for source agent il and denoted by r^{il}, such that

$$r^{il}=r^{il}(x^{il}), \quad \forall i,l, \tag{3}$$

where r^{il} is assumed to be strictly convex and continuously differentiable. Note that the risk function in (3) is dependent both on the financial transactions conducted physically as well as electronically. Clearly, a possible risk function could be constructed as follows. Assume a variance-covariance matrix Q^{il} associated with agent il, which is of dimension $(2JH+KHL)\times(2JH+KHL)$, symmetric, and positive definite. Then a possible risk function for source agent i in country l would be given by:

$$r^{il}(x^{il}) = x^{il^T} Q^{il} x^{il}, \quad \forall i, l. \tag{4}$$

In such a case, one assumes that each source agent's uncertainty, or assessment of risk, is based on a variance-covariance matrix representing the source agent's assessment of the standard deviation of the prices of the financial instruments in the distinct currencies (see also Markowitz (1952, 1959)).

Hence, the second criterion of source agent il corresponding to risk minimization can be expressed as:

$$\text{Minimize} \quad r^{il}(x^{il}). \tag{5}$$

We are now ready to construct the multicriteria decision-making problem for a source agent in a particular country.

The Multicriteria Decision-Making Problem for a Source Agent in a Particular Country

Each source agent il associates a nonnegative weight α^{il} with the risk minimization criterion (5), with the weight associated with the net revenue maximization criterion (2) serving as the numeraire and being set equal to 1. Thus, we can construct a *value function* for each source agent (cf. Fishburn (1970), Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993)) using a constant additive weight value function. Consequently, the multicriteria decision-making problem for source agent il (for which the constraints are also included) with the multicriteria objective function denoted by U^{il} can be transformed into:

$$\begin{aligned} \text{Maximize} \quad U^{il}(x^{il}) = & \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho_{1jhm}^{il*} + e_h^*)x_{jhm}^{il} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho_{1kh\hat{l}}^{il*} + e_h^*)x_{kh\hat{l}}^{il} \\ & -\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} c_{jhm}^{il}(x_{jhm}^{il}) - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} c_{kh\hat{l}}^{il}(x_{kh\hat{l}}^{il}) - \alpha^{il} r^{il}(x^{il}), \end{aligned} \tag{6}$$

subject to:

$$\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} x_{jhm}^{il} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} x_{kh\hat{l}}^{il} \le S^{il}, \tag{7}$$

and the nonnegativity assumption:

$$x_{jhm}^{il} \geq 0, \quad x_{kh\hat{l}}^{il} \geq 0, \quad \forall j, h, m, k, \hat{l}, \tag{8}$$

that is, according to (7), the allocations of source agent il's funds among those available from the different intermediaries in distinct currencies and transacted electronically with the consumers at the demand markets cannot exceed his holdings. The first two terms in the objective function (6) denote the revenue whereas the third and fourth terms denote the transaction costs and the last term denotes the weighted risk. Note also that the objective function given in (6) is strictly concave in the x^{il} variables. Constraint (7) allows a source agent not to invest a portion (or all) of his funds, with the "slack," that is, the funds not invested by agent i in country l being given by s^{il}.

Optimality Conditions for All Source Agents

The optimality conditions of all source agents i; $i = 1, \dots, I$; in all countries: l; $l = 1, \dots, L$ (see also Bazaraa, Sherali, and Shetty (1993), Bertsekas and Tsitsiklis (1992), Nagurney and Cruz (2002)), under the above stated assumptions on the underlying functions, can be expressed as: determine $(x^{1*}, x^{2*}) \in \mathcal{K}^1$, satisfying

$$\sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{j=1}^{J} \sum_{h=1}^{H} \sum_{m=1}^{2} \left[\alpha^{il} \frac{\partial r^{il}(x^{il*})}{\partial x_{jhm}^{il}} + \frac{\partial c_{jhm}^{il}(x_{jhm}^{il*})}{\partial x_{jhm}^{il}} - \rho_{1jhm}^{il*} \; - e_h^* \right] \times \left[x_{jhm}^{il} - x_{jhm}^{il*} \right]$$

$$+ \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{j=1}^{J} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \left[\alpha^{il} \frac{\partial r^{il}(x^{il*})}{\partial x_{kh\hat{l}}^{il}} + \frac{\partial c_{kh\hat{l}}^{il}(x_{kh\hat{l}}^{il*})}{\partial x_{kh\hat{l}}^{il}} - \rho_{1kh\hat{l}}^{il*} \; - e_h^* \right] \times \left[x_{kh\hat{l}}^{il} - x_{kh\hat{l}}^{il*} \right] \geq 0,$$

$$\forall (x^1, x^2) \in \mathcal{K}^1, \tag{9}$$

where $\mathcal{K}^1 \equiv \{(x^1, x^2) | (x^1, x^2) \in R_+^{IL(2JH+KHL)}$ and satisfies (7), $\forall i, l\}$.

The Behavior of the Intermediaries and their Optimality Conditions

The intermediaries (cf. Figure 1), in turn, are involved in transactions both with the source agents in the different countries, as well as with the users of the funds, that is, with the ultimate consumers associated with the markets for the distinct types of loans/products in different currencies and countries and represented by the bottom tier of nodes of the network.

Thus, an intermediary conducts transactions via a physical link, and/or electronically via an Internet link both with the source agents as well as with the consumers at the demand markets.

An intermediary j is faced with what we term a *handling/conversion* cost, which may include, for example, the cost of converting the incoming financial flows into the financial loans/products associated with the demand markets. We denote such a cost faced by intermediary j by c_j and, in the simplest case, c_j would be a function of $\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} x^{il}_{jhm}$, that is, the holding/conversion cost of an intermediary is a function of how much he has obtained in the different currencies from the various source agents in the different countries. For the sake of generality, however, we allow the function to depend also on the amounts held by other intermediaries and, therefore, we may write:

$$c_j = c_j(x^1), \quad \forall j. \tag{10}$$

The intermediaries also have associated transaction costs in regards to transacting with the source agents, which can depend on the type of currency as well as the source agent. We denote the transaction cost associated with intermediary j transacting with agent il associated with currency h via mode m by $\hat{c}^{il}_{jhm}$ and we assume that it is of the form

$$\hat{c}^{il}_{jhm} = \hat{c}^{il}_{jhm}(x^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{11}$$

that is, such a transaction cost is allowed to depend on the amount allocated by the particular agent in a currency and transacted with the particular intermediary via the particular mode. In addition, we assume that an intermediary j also incurs a transaction cost $c^j_{kh\hat{l}m}$ associated with transacting with demand market $kh\hat{l}$, where

$$c^j_{kh\hat{l}m} = c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m}), \quad \forall j, k, h, \hat{l}, m. \tag{12}$$

Hence, the transaction costs given in (12) can vary according to the intermediary/product/currency/country combination and are a function of the volume of the product transacted. We assume that the cost functions (10) – (12) are convex and continuously differentiable and that the costs are measured in the base currency.

The actual price charged for the financial product k associated with intermediary j transacting with the consumers in currency h via mode m and country $\hat{l}$ is denoted by $\rho^{j*}_{2kh\hat{l}m}$, for intermediary j. Later, we discuss how such prices are arrived at.

We assume that each intermediary seeks to maximize his net revenue with the net revenue criterion for intermediary j being given by:

$$\text{Maximize} \quad \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}(\rho^{j*}_{2kh\hat{l}m}+e^*_h)y^j_{kh\hat{l}m} - c_j(x^1) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\hat{c}^{il}_{jhm}(x^{il}_{jhm})$$

$$-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m}) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho^{il*}_{1jhm}+e^*_h)x^{il}_{jhm}. \tag{13}$$

The first term in (13) represents the revenue of intermediary j; the subsequent three terms correspond to the costs with the fifth term denoting the payments to the source agents for the funds.

We assume that the intermediaries are also concerned with risk minimization and that they have risk associated both with transacting with the various source agents in the different countries and with the consumers for the products in the different currencies and countries. The risk can also depend on the mode of transaction. Hence, we assume for each intermediary j a risk function r^j, which is strictly convex in its variables and continuously differentiable, and of the form:

$$r^j = r^j(x^1, y), \quad \forall j. \tag{14}$$

For example, the risk for intermediary j could be represented by a variance-covariance matrix denoted by Q^j with this matrix being positive definite and of dimensions $(2IL + 2KHL) \times (2IL + 2KHL)$ for each intermediary j. Such a matrix would reflect the risk associated with transacting with the various source agents in the different countries and with the consumers at the demand markets for the products in different currencies and in different countries. If we let x_j, without any loss in generality, denote the $2ILH$-dimensional column vector with the $ilhm$-th component given by x^{il}_{jhm}. Indeed, then a possible risk function for intermediary j could be represented by the function:

$$r^j(x^1, y) = \begin{bmatrix} x_j \\ y^j \end{bmatrix}^T Q^j \begin{bmatrix} x_j \\ y^j \end{bmatrix}. \tag{15}$$

Note that, for the sake of modeling generality and flexibility, we allow the risk function for an intermediary to depend not only on the financial flows flowing "into" and "out of" that intermediary but on the other financial flows as well. The risk function given by (15) is actually a special case of the one in (14) in that it depends only on the financial volumes that the particular intermediary actually deals with.

The Multicriteria Decision-Making Problem for an Intermediary

We assume that each intermediary j associates a weight of 1 with his net revenue criterion and a weight of β^j with his risk level, with this term being nonnegative. Therefore, akin to the mechanism used for each source agent, we have that the multicriteria decision-making problem for each intermediary can be transformed directly into the optimization problem, with the multicriteria objective function denoted by U^j, and the decision-maker also faced with constraints. Hence, we may write:

$$\text{Maximize} \quad U^j(x_j, y^j) = \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \sum_{m=1}^{2} (\rho^{j*}_{2kh\hat{l}m} + e^*_h) y^j_{kh\hat{l}m} - c_j(x^1) - \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{2} \hat{c}^{il}_{jhm}(x^{il}_{jhm})$$

$$- \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \sum_{m=1}^{2} c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m}) - \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{2} (\rho^{il*}_{1jhm} + e^*_h) x^{il}_{jhm} - \beta^j r^j(x^1, y) \tag{16}$$

subject to:

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \sum_{m=1}^{2} y^j_{kh\hat{l}m} \leq \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{2} x^{il}_{jhm} \tag{17}$$

and the nonnegativity constraints:

$$x^{il}_{jhm} \geq 0, \quad y^j_{kh\hat{l}m} \geq 0, \quad \forall i, l, h, \hat{l}, m. \tag{18}$$

The multicriteria objective function (16) expresses that the difference between the revenues (given by the first term) minus the handling cost, the two sets of transaction costs, and the payout to the source agents (given by the subsequent four terms, respectively) should be maximized, whereas the weighted risk (see the last term in (16)) should be minimized. The objective function in (16) is concave in its variables under the above imposed assumptions.

Here we assume that the financial intermediaries can compete, with the governing optimality/equilibrium concept underlying noncooperative behavior being that of Nash (1950, 1951), which states that each decision-maker (intermediary) will determine his optimal strategies, given the optimal ones of his competitors. The optimality conditions for all financial intermediaries simultaneously, under the above stated assumptions, can be compactly expressed as (cf. Gabay and Moulin (1980), Dafermos and Nagurney (1987), and Nagurney and Ke (2001, 2003)): determine $(x^{1*}, y^*, \gamma^*) \in R_+^{2ILJH+2JKHL+J}$, such that

$$\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\beta^j\frac{\partial r^j(x^{1*},y^*)}{\partial x^{il}_{jhm}}+\frac{\partial c_j(x^{1*})}{\partial x^{il}_{jhm}}+\rho^{il*}_{1jhm}+e^*_h+\frac{\partial \hat{c}^{il}_{jhm}(x^{il*}_{jhm})}{\partial x^{il}_{jhm}}-\gamma^*_j\right]\times\left[x^{il}_{jhm}-x^{il*}_{jhm}\right]$$

$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\beta^j\frac{\partial r^j(x^{1*},y^*)}{\partial y^j_{kh\hat{l}m}}+\frac{\partial c^j_{khl}(y^{j*}_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}}-\rho^{j*}_{2kh\hat{l}m}-e^*_h+\gamma^*_j\right]\times\left[y^j_{kh\hat{l}m}-y^{j*}_{kh\hat{l}m}\right]$$

$$+\sum_{j=1}^{J}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}x^{il*}_{jhm}-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}y^{j*}_{kh\hat{l}m}\right]\times\left[\gamma_j-\gamma^*_j\right]\geq 0,$$

$$\forall (x^1, y, \gamma) \in R_+^{2ILJH+2JKHL+J}, \tag{19}$$

where γ_j is the Lagrange multiplier associated with constraint (17) (see Bazaraa, Sherali, and Shetty (1993)), and γ is the J-dimensional column vector of Lagrange multiplers of all the intermediaries with γ^* denoting the vector of optimal multipliers.

The Consumers at the Demand Markets and the Equilibrium Conditions

We now describe the consumers located at the demand markets. The consumers take into account in making their consumption decisions not only the price charged for the financial product by the agents with source of funds and intermediaries but also their transaction costs associated with obtaining the product.

Let $\hat{c}^j_{kh\hat{l}m}$ denote the transaction cost associated with obtaining product k in currency h in country $\hat{l}$ via mode m from intermediary j and recall that $y^j_{kh\hat{l}m}$ is the amount of the financial product k in currency h flowing between intermediary j and consumers in country $\hat{l}$ via mode m. We assume that the transaction cost is measured in the base currency, is continuous, and of the general form:

$$\hat{c}^j_{kh\hat{l}m}=\hat{c}^j_{kh\hat{l}m}(y), \quad \forall j,k,h,\hat{l},m. \tag{20a}$$

Furthermore, let $\hat{c}^{il}_{kh\hat{l}}$ denote the transaction cost associated with obtaining the financial product k in currency h in country $\hat{l}$ electronically from source agent il, where we assume that the transaction cost is continuous and of the general form:

$$\hat{c}^{il}_{kh\hat{l}} = \hat{c}^{il}_{kh\hat{l}}(x^2), \quad \forall i, l, k, h, \hat{l}. \tag{20b}$$

Hence, the transaction cost associated with transacting directly with source agents is of a form of the same level of generality as the transaction costs associated with transacting with the financial intermediaries.

Denote the demand for product k in currency h in country $\hat{l}$ by $d_{kh\hat{l}}$ and assume, as given, the continuous demand functions:

$$d_{kh\hat{l}} = d_{kh\hat{l}}(\rho_3), \quad \forall k, h, \hat{l}. \tag{21}$$

Thus, according to (21), the demand of consumers for the financial product in a currency and country depends, in general, not only on the price of the product at that demand market (and currency and country) but also on the prices of the other products at the other demand markets (and in other countries and currencies). Consequently, consumers at a demand market, in a sense, also compete with consumers at other demand markets.

The consumers take the price charged by the intermediary, which was denoted by $\rho^{j*}_{2kh\hat{l}m}$ for intermediary j, product k, currency h, and country $\hat{l}$ via mode m, the price charged by source agent il, which was denoted by $\rho^{il*}_{1kh\hat{l}}$, and the rate of appreciation in the currency, plus the transaction costs, in making their consumption decisions. The equilibrium conditions for the consumers at demand market $kh\hat{l}$, thus, take the form: for all intermediaries: $j = 1, \ldots, J$ and all mode m; $m = 1, 2$:

$$\rho^{j*}_{2kh\hat{l}m} + e^*_h + \hat{c}^{j}_{kh\hat{l}m}(y^*) \begin{cases} = \rho^*_{3kh\hat{l}}, & \text{if} \quad y^{j*}_{kh\hat{l}m} > 0 \\ \geq \rho^*_{3kh\hat{l}}, & \text{if} \quad y^{j*}_{kh\hat{l}m} = 0, \end{cases} \tag{22}$$

and for all source agents il; $i = 1, \ldots, I$ and $l = 1, \ldots, L$:

$$\rho^{il*}_{1kh\hat{l}} + e^*_h + \hat{c}^{il}_{kh\hat{l}}(x^{2*}) \begin{cases} = \rho^*_{3kh\hat{l}}, & \text{if} \quad x^{il*}_{kh\hat{l}} > 0 \\ \geq \rho^*_{3kh\hat{l}}, & \text{if} \quad x^{il*}_{kh\hat{l}} = 0, \end{cases} \tag{23}$$

In addition, we must have that

$$d_{kh\hat{l}}(\rho_3^*) \begin{cases} = \sum_{j=1}^{J}\sum_{m=1}^{2} y_{kh\hat{l}m}^{j*} + \sum_{i=1}^{I}\sum_{l=1}^{L} x_{kh\hat{l}}^{il*}, & \text{if} \quad \rho_{3kh\hat{l}}^* > 0 \\ \leq \sum_{j=1}^{J}\sum_{m=1}^{2} y_{kh\hat{l}m}^{j*} + \sum_{i=1}^{I}\sum_{l=1}^{L} x_{kh\hat{l}}^{il*}, & \text{if} \quad \rho_{3kh\hat{l}}^* = 0. \end{cases} \tag{24}$$

Conditions (22) state that consumers at demand market $kh\hat{l}$ will purchase the product from intermediary j, if the price charged by the intermediary for the product and the appreciation rate for the currency plus the transaction cost (from the perspective of the consumer) does not exceed the price that the consumers are willing to pay for the product in that currency and country, i.e., $\rho_{3kh\hat{l}}^*$. Note that, according to (22), if the transaction costs are identically equal to zero, then the price faced by the consumers for a given product is the price charged by the intermediary for the particular product and currency in the country plus the rate of appreciation in the currency. Conditions (23) state the analogue, but for the case of electronic transactions with the source agents.

Condition (24), on the other hand, states that, if the price the consumers are willing to pay for the financial product at a demand market is positive, then the quantity of at the demand market is precisely equal to the demand.

In equilibrium, conditions (22), (23), and (24) will have to hold for all demand markets and these, in turn, can be expressed also as an inequality analogous to those in (9) and (19) and given by: determine $(x^{2*}, y^*, \rho_3^*) \in R_+^{(IL+2J+1)KHL}$, such that

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\rho_{2kh\hat{l}m}^{j*} + e_h^* + \hat{c}_{kh\hat{l}m}^{j}(y^*) - \rho_{3kh\hat{l}}^*\right] \times \left[y_{kh\hat{l}m}^{j} - y_{kh\hat{l}m}^{j*}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\rho_{1kh\hat{l}}^{il*} + e_h^* + \hat{c}_{kh\hat{l}}^{il}(x^{2*}) - \rho_{3kh\hat{l}}^*\right] \times \left[x_{kh\hat{l}}^{il} - x_{kh\hat{l}}^{il*}\right]$$

$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\sum_{j=1}^{J}\sum_{m=1}^{2} y_{kh\hat{l}m}^{j*} + \sum_{i=1}^{I}\sum_{l=1}^{L} x_{kh\hat{l}}^{il*} - d_{kh\hat{l}}(\rho_3^*)\right] \times \left[\rho_{3kh\hat{l}} - \rho_{3kh\hat{l}}^*\right] \geq 0,$$

$$\forall (x^2, y, \rho_3) \in R_+^{(IL+2J+1)KHL}. \tag{25}$$

The Equilibrium Conditions for the International Financial Network with Multicriteria Decision-Makers

In equilibrium, the financial flows that the source agents in different countries transact with the intermediaries must coincide with those that the intermediaries actually accept from them. In addition, the amounts of the financial products that are obtained by the consumers in the different countries and currencies must be equal to the amounts that both the source agents and the intermediaries actually provide. Hence, although there may be competition between decision-makers at the same level of tier of nodes of the financial network there must be, in a sense, cooperation between decision-makers associated with pairs of nodes (through positive flows on the links joining them). Thus, in equilibrium, the prices and financial flows must satisfy the sum of the optimality conditions (9) and (19) and the equilibrium conditions (25). We make these relationships rigorous through the subsequent definition and variational inequality derivation below.

Definition 1: International Financial Network Equilibrium with Multicriteria Decision-Makers

The equilibrium state of the international financial network with multicriteria decision-makers and with electronic transactions is one where the financial flows between the tiers of the network coincide and the financial flows and prices satisfy the sum of conditions (9), (19), and (25).

The equilibrium state is equivalent to the following:

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the international financial network with multicriteria decision-makers and with electronic transactions according to Definition 1 are equivalent to the solution of the variational inequality given by: determine $(x^{1*}, x^{2*}, y^*, \gamma^*, \rho_3^*) \in \mathcal{K}$*, satisfying:*

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\alpha^{il}\frac{\partial r^{il}(x^{il*})}{\partial x^{il}_{jhm}} + \frac{\partial c^{il}_{jhm}(x^{il*}_{jhm})}{\partial x^{il}_{jhm}} + \beta^{j}\frac{\partial r^{j}(x^{1*}, y^*)}{\partial x^{il}_{jhm}} + \frac{\partial c_j(x^{1*})}{\partial x^{il}_{jhm}} + \frac{\partial \hat{c}^{il}_{jhm}(x^{il*}_{jhm})}{\partial x^{il}_{jhm}} - \gamma^*_j\right]$$

$$\times \left[x_{jhm}^{il} - x_{jhm}^{il*}\right] + \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\alpha^{il}\frac{\partial r^{il}(x^{il*})}{\partial x_{kh\hat{l}}^{il}} + \frac{\partial c_{kh\hat{l}}^{il}(x_{kh\hat{l}}^{il*})}{\partial x_{kh\hat{l}}^{il}} + \hat{c}_{kh\hat{l}}^{il}(x^{2*}) - \rho_{3kh\hat{l}}^{*}\right] \times \left[x_{kh\hat{l}}^{il} - x_{kh\hat{l}}^{il*}\right]$$

$$+ \sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\beta^{j}\frac{\partial r^{j}(x^{1*}, y^{*})}{\partial y_{kh\hat{l}m}^{j}} + \frac{\partial c_{khlm}^{j}(y_{kh\hat{l}m}^{j*})}{\partial y_{kh\hat{l}m}^{j}} + \hat{c}_{kh\hat{l}m}^{j}(y^{*}) - \rho_{3kh\hat{l}}^{*} + \gamma_{j}^{*}\right] \times \left[y_{kh\hat{l}m}^{j} - y_{kh\hat{l}m}^{j*}\right]$$

$$+ \sum_{j=1}^{J}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} x_{jhm}^{il*} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} y_{kh\hat{l}m}^{j*}\right] \times \left[\gamma_{j} - \gamma_{j}^{*}\right]$$

$$+ \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\sum_{j=1}^{J}\sum_{m=1}^{2} y_{kh\hat{l}m}^{j*} + \sum_{i=1}^{I}\sum_{l=1}^{L} x_{kh\hat{l}}^{il*} - d_{kh\hat{l}}(\rho_{3}^{*})\right] \times \left[\rho_{3kh\hat{l}} - \rho_{3kh\hat{l}}^{*}\right] \geq 0,$$

$$\forall (x^{1}, x^{2}, y, \gamma, \rho_{3}) \in \mathcal{K}, \tag{26}$$

where $\mathcal{K} \equiv \{\mathcal{K}^{1} \times \mathcal{K}^{2}\}$, *and* $\mathcal{K}^{2} \equiv \{(y, \gamma, \rho_{3}) | (y, \gamma, \rho_{3}) \in R_{+}^{2JKHL+J+KHL}\}$.

Proof: Summation of inequalities (9), (19), and (25), yields, after algebraic simplification, the variational inequality (26). □

We now put variational inequality (26) into standard form which will be utilized in the subsequent sections. For additional background on variational inequalities and their applications, see the book by Nagurney (1999). In particular, we have that variational inequality (26) can be expressed as:

$$\langle F(X^{*}), X - X^{*} \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{27}$$

where $X \equiv (x^{1}, x^{2}, y, \gamma, \rho_{3})$ and $F(X) \equiv (F_{iljhm}, F_{ilkh\hat{l}}, F_{jkh\hat{l}m}, F_{j}, F_{kh\hat{l}})$ with indices: $i = 1, \ldots, I$; $\hat{l} = 1, \ldots, L$; $j = 1, \ldots, J$; $h = 1, \ldots, H$; $m = 1, 2$, and the specific components of F given by the functional terms preceding the multiplication signs in (26), respectively. The term $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space.

We now describe how to recover the prices associated with the first two tiers of nodes in the international financial network. Clearly, the components of the vector ρ_{3}^{*} are obtained directly from the solution of variational inequality (26) as will be demonstrated explicitly through several numerical examples in Section 6. In order to recover the second tier prices associated with the intermediaries and the exchange rates one can (after solving variational inequality (26) for the particular numerical problem) *either* (cf. (22)) set $\rho_{2kh\hat{l}m}^{j*} + e_{h}^{*} =$

$\rho^*_{3kh\hat{l}} - \hat{c}_{kh\hat{l}}(y^*)$, for any $j, k, h, \hat{l}, m$ such that $y^{j*}_{kh\hat{l}m} > 0$, *or* (cf. (19)) for any $y^{j*}_{kh\hat{l}m} > 0$, set $\rho^{j*}_{2kh\hat{l}m} + e^*_h = \beta^j \frac{\partial r^j(x^{1*}, y^*)}{\partial y^j_{kh\hat{l}m}} + \frac{\partial c^j_{kh\hat{l}m}(y^{j*}_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} - \gamma^*_j$.

Similarly, from (19) we can infer that the top tier prices comprising the vector ρ^*_1 can be recovered (once the variational inequality (26) is solved with particular data) thus: for any i, l, j, h, m, such that $x^{il*}_{jhm} > 0$, set $\rho^{il*}_{1jhm} + e^*_h = \gamma^*_j - \beta^j \frac{\partial r^j(x^{1*}, y^*)}{\partial x^{il}_{jhm}} - \frac{\partial c_j(x^{1*})}{\partial x^{il}_{jhm}} - \frac{\partial \hat{c}^{il}_{jhm}(x^{il*}_{jhm})}{\partial x^{il}_{jhm}}$.

In addition, in order to recover the first tier prices associated with the demand market and the exchange rates one can (after solving variational inequality (26) for the particular numerical problem) *either* (cf. (23)) set $\rho^{il*}_{1kh\hat{l}} + e^*_h = \rho^*_{3kh\hat{l}} - \hat{c}^{il}_{kh\hat{l}}(x^{2*})$, for any $i, l, k, h, \hat{l}$ such that $x^{il*}_{kh\hat{l}} > 0$, *or* (cf. (9)) for any $x^{il*}_{kh\hat{l}} > 0$, set $\rho^{il*}_{1kh\hat{l}} + e^*_h = \alpha^{il} \frac{\partial r^{il}(x^{il*})}{\partial x^{il}_{kh\hat{l}}} + \frac{\partial c^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}}$

Note that in the absence of electronic transactions and with the weights associated with risk all being equal to 1 the above model collapses to the model developed by Nagurney and Cruz (2002).

3. The Dynamic Adjustment Process

In this Section, we propose a dynamic adjustment process which describes the disequilibrium dynamics as the various international financial network decision-makers adjust their financial flows between the tiers and the prices associated with the different tiers adjust as well. Importantly, the set of stationary points of the projected dynamical system which formulates the dynamic adjustment process will coincide with the set of solutions to the variational inequality problem (26).

We begin by describing the dynamics underlying the prices of the financial products at the various demand markets. We then proceed upward through the international financial network (cf. Figure 1) to propose the dynamics of the financial flows and that of the prices associated with the second tier of nodes.

Demand Market Price Dynamics

We assume that the rate of change of the price $\rho_{3kh\hat{l}}$, denoted by $\dot{\rho}_{3kh\hat{l}}$, is equal to the difference between the demand for the financial product at the demand market and the amount of the product actually available at that particular market. Hence, if the demand for the product at the demand market at an instant in time exceeds the amount available from the various intermediaries and source agents, then the price will increase; if the amount available exceeds the demand at the price, then the price will decrease. Moreover, it is guaranteed that the prices do not become negative. Thus, the dynamics of the price $\rho_{3kh\hat{l}}$ for each $k, h, \hat{l}$ can be expressed as:

$$\dot{\rho}_{3kh\hat{l}} = \begin{cases} d_{kh\hat{l}}(\rho_3) - \sum_{j=1}^{J}\sum_{m=1}^{2} y^j_{kh\hat{l}m} - \sum_{i=1}^{I}\sum_{l=1}^{L} x^{il}_{kh\hat{l}}, & \text{if} \quad \rho_{3kh\hat{l}} > 0 \\ \max\{0, d_{kh\hat{l}}(\rho_3) - \sum_{j=1}^{J}\sum_{m=1}^{2} y^j_{kh\hat{l}m} - \sum_{i=1}^{I}\sum_{l=1}^{L} x^{il}_{kh\hat{l}}\}, & \text{if} \quad \rho_{3kh\hat{l}} = 0. \end{cases} \tag{28}$$

The Dynamics of the Financial Products between the Intermediaries and the Demand Markets

The rate of change of the financial flow $y^j_{kh\hat{l}m}$, in turn, and denoted by $\dot{y}^j_{kh\hat{l}m}$, is assumed to be equal to the difference between the price the consumers are willing to pay for the financial product at the demand market minus the price charged and the various transaction costs and the weighted marginal risk associated with the transaction. Here we also guarantee that the financial flows do not become negative. Hence, we may write: for every $j, k, h, \hat{l}, m$:

$$\dot{y}^j_{kh\hat{l}m} = \begin{cases} \rho_{3kh\hat{l}} - \frac{\partial c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} - \hat{c}^j_{kh\hat{l}m}(y) - \beta^j \frac{\partial r^j(x^1,y)}{\partial y_{kh\hat{l}m}} - \gamma_j, & \text{if} \quad y^j_{kh\hat{l}m} > 0 \\ \max\{0, \rho_{3kh\hat{l}} - \frac{\partial c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} - \hat{c}^j_{kh\hat{l}m}(y) - \beta^j \frac{\partial r^j(x^1,y)}{\partial y^j_{kh\hat{l}m}} - \gamma_j\}, & \text{if} \quad y^j_{kh\hat{l}m} = 0. \end{cases} \tag{29}$$

Hence, according, to (29), if the price that the consumers are willing to pay for the product (in the currency and country) exceeds the price that the intermediary charges and the various transaction costs and weighted marginal risk, then the volume of flow of the product to that demand market will increase; otherwise, it will decrease (or remain unchanged).

The Dynamics of the Prices at the Intermediaries

The prices at the intermediaries, whether they are physical or virtual, must reflect supply and demand conditions as well. In particular, we let $\dot{\gamma}_j$ denote the rate of change in the market clearing price associated with intermediary j and we propose the following dynamic adjustment for every intermediary j:

$$\dot{\gamma}_j = \begin{cases} \sum_{k=1}^K \sum_{h=1}^H \sum_{\hat{l}=1}^L \sum_{m=1}^2 y^j_{kh\hat{l}m} - \sum_{i=1}^I \sum_{l=1}^L \sum_{h=1}^H \sum_{m=1}^2 x^{il}_{jhm}, & \text{if} \quad \gamma_j > 0 \\ \max\{0, \sum_{k=1}^K \sum_{h=1}^H \sum_{\hat{l}=1}^L \sum_{m=1}^2 y^j_{kh\hat{l}m} - \sum_{i=1}^I \sum_{l=1}^L \sum_{h=1}^H \sum_{m=1}^2 x^{il}_{jhm}\}, & \text{if} \quad \gamma_j = 0. \end{cases} \tag{30}$$

Hence, if the financial flows into an intermediary exceed the amount demanded at the demand markets from the intermediary, then the market-clearing price at that intermediary will decrease; if, on the other hand, the volume of financial flows into an intermediary is less than that demanded by the consumers at the demand markets (and handled by the intermediary), then the market-clearing at that intermediary price will increase.

The Dynamics of the Financial Flows from the Source Agents

Note that, unlike the financial flows (as well as the prices associated with the distinct nodal tiers of the network) between the intermediaries and the demand markets, the financial flows from the source agents are subject not only to nonnegativity constraints but also to budget constraints (cf. (7)). Hence, in order to guarantee that these constraints are not violated we need to introduce some additional machinery based on projected dynamical systems theory in order to describe the dynamics of these financial flows (see also, e.g., Nagurney and Zhang (1996a) and Nagurney, Cruz, and Matsypura (2003)).

In particular, we denote the rate of change of the vector of financial flows from source agent il by $\dot{x}^{il}$ and noting that the *best realizable direction* for the financial flows from source agent il must include the constraints, we have that:

$$\dot{x}^{il} = \Pi_{\mathcal{K}^{il}}(x^{il}, -F^{il}), \tag{31}$$

where Π_K is defined as:

$$\Pi_K(x, v) = \lim_{\delta \to 0} \frac{P_K(x + \delta v) - x}{\delta}, \tag{32}$$

and P_K is the norm projection defined by

$$P_K(x) = \text{argmin}_{x' \in K} \|x' - x\|. \tag{33}$$

The feasible set $\mathcal{K}^{il}$ is defined as: $\mathcal{K}^{il} \equiv \{x^{il} | x^{il} \in R_+^{2JH+KHL}$ and satisfies (7)$\}$, and F^{il} is the vector (see following (27)) with components: $F_{iljhm}, F_{ilkh\hat{l}}$ and with indices: $j = 1, \ldots, J$; $h = 1, \ldots, H$; $m = 1, 2$, and $k = 1, \ldots, K$. Hence, expression (31) reflects that the financial flow on a link emanating from a source agent will increase if the price (be it the market-clearing price associated with an intermediary or a demand market price) exceeds the various costs and weighted marginal risk; it will decrease if the latter exceeds the former.

The Projected Dynamical System

Consider now a dynamical system in which the demand market prices evolve according to (28); the financial flows between intermediaries and the demand markets evolve according to (29); the prices at the intermediaries evolve according to (30), and the financial flows from the source agents evolve according to (31) for all source agents il. Let X and $F(X)$ be as defined following (27) and recall the feasible set $\mathcal{K}$. Then the dynamic model described by (28)–(31) can be rewritten as a *projected dynamical system* (Nagurney and Zhang (1996a)) defined by the following initial value problem:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0, \tag{34}$$

where $\Pi_{\mathcal{K}}$ is the projection operator of $-F(X)$ onto $\mathcal{K}$ at X (cf. (32)) and $X_0 = (x^{10}, x^{20}, y^0, \gamma^0, \rho_3^0)$ is the initial point corresponding to the initial financial flow and price pattern.

The trajectory of (34) describes the dynamic evolution of and the dynamic interactions among the international financial flows and prices. The dynamical system (34) is non-classical since it has a discontinuous right-hand side due to the projection operation. Such dynamical systems were introduced by Dupuis and Nagurney (1993) and have been used to study a plethora of dynamic models in economic, finance, and transportation (see Nagurney and Zhang (1996a)). In addition, the projected dynamical systems methodology has been used to-date to formulate dynamical supply chain network models, with and without electronic commerce (see, e.g., Nagurney and Dong (2002), Nagurney, Cruz, and Matsypura (2003), and the references therein).

Here we apply the methodology, for the first time, to international financial network modelling, analysis, and computation in the case of intermediation, electronic transactions, and multicriteria decision-making. The following result is immediate from Dupuis and Nagurney (1993).

Theorem 2: Set of Stationary Points Coincides with Set of Equilibrium Points

The set of stationary points of the projected dynamical system (34) coincides with the set of solutions of the variational inequality problem (27) and, thus, with the set of equilibrium points as defined in Definition 1.

With Theorem 2, we see that the dynamical system proposed in this Section, provides the disequilibrium dynamics prior to the steady or equilibrium state of the international financial network. Hence, once, a stationary point of the projected dynamical system is reached, that is, when $\dot{X} = 0$ in (34), that point (consisting of financial flows and prices) also satisfies variational inequality (27); equivalently, (26), and is, therefore, an international financial network equilibrium according to Definition 1.

The above described dynamics are very reasonable from an economic perspective and also illuminate that there must be cooperation between tiers of decision-makers although there may be competition within a tier.

4. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (26). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (27)) that enters the variational inequality of interest here. Finally, we establish that the trajectories of the projected dynamical system (34) are well-defined under reasonable assumptions.

Since the feasible set is not compact we cannot derive existence simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern. Let

$$\mathcal{K}_b \equiv \{(x^1, x^2, y, \gamma, \rho_3) | 0 \leq x^1 \leq b_1;\ 0 \leq x^2 \leq b_1; 0 \leq y \leq b_3; 0 \leq \gamma \leq b_4; 0 \leq \rho_3 \leq b_5\}, \quad (35)$$

where $b = (b_1, b_2, b_3, b_4, b_5) \geq 0$ and $x^1 \leq b_1; x^2 \leq b_2; y \leq b_3; \gamma \leq b_4; \rho_3 \leq b_5$ means that $x^{il}_{jhm} \leq b_1; x^{il}_{kh\hat{l}} \leq b_2$; $y^j_{kh\hat{l}m} \leq b_3$; $\gamma_j \leq b_4$; and $\rho_{3kh\hat{l}} \leq b_5$ for all $i, l, j, k, h, \hat{l}, m$. Then $\mathcal{K}_b$ is a bounded closed convex subset of $R^{2ILJH+ILKHL+2JKHL+J+KHL}$. Thus, the following variational inequality

$$\langle F(X^b), X - X^b \rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \quad (36)$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, since $\mathcal{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

Theorem 3

Variational inequality (27) admits a solution if and only if there exists a $b > 0$, such that variational inequality (36) admits a solution in $\mathcal{K}_b$ with

$$x^{1b} < b_1, \quad x^{2b} < b_2, \quad y^b < b_3, \quad \gamma^b < b_4, \quad \rho_3^b < b_5. \quad (37)$$

Theorem 4: Existence of a Solution

Suppose that there exist positive constants M, N, R, with $R > 0$, such that:

$$\alpha^{il}\frac{\partial r^{il}(x^{il})}{\partial x^{il}_{jhm}} + \frac{\partial c^{il}_{jhm}(x^{il}_{jhm})}{\partial x^{il}_{jhm}} + \beta^{j}\frac{\partial r^{j}(x^1, y)}{\partial x^{il}_{jhm}} + \frac{\partial c_j(x)}{\partial x^{il}_{jhm}} + \frac{\partial \hat{c}^{il}_{jhm}(x^{il}_{jhm})}{\partial x^{il}_{jhm}} \geq M,$$

$$\forall x^1 \text{ with } x^{il}_{jhm} \geq N, \quad \forall i, l, j, h, m, \tag{38}$$

$$\alpha^{il}\frac{\partial r^{il}(x^{il})}{\partial x^{il}_{kh\hat{l}}} + \frac{\partial c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}} + \hat{c}^{il}_{kh\hat{l}}(x^2) \geq M, \forall x^2 \text{ with } x^{il}_{kh\hat{l}} \geq N, \quad \forall i, l, j, h, \hat{l}, \tag{39}$$

$$\beta^{j}\frac{\partial r^{j}(x^1, y)}{\partial y^{j}_{kh\hat{l}m}} + \frac{\partial c^{j}_{khl}(y^{j}_{kh\hat{l}m})}{\partial y^{j}_{kh\hat{l}m}} + \hat{c}^{j}_{kh\hat{l}m}(y) \geq M, \quad \forall y \text{ with } y^{j}_{kh\hat{l}m} \geq N, \quad \forall j, k, h, \hat{l}, m, \tag{40}$$

$$d_{kh\hat{l}}(\rho_3^*) \leq N, \quad \forall \rho_3 \text{ with } \rho_{3kh\hat{l}} > R, \quad \forall k, h, \hat{l}. \tag{41}$$

Then variational inequality (26); equivalently, variational inequality (27), admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993) (see also the existence proof in Nagurney and Ke (2001)). □

Assumptions (38), (39), and (40) are reasonable from an economics perspective, since when the financial flow between a source agent and intermediary or demand market or between an intermediary and demand market is large, we can expect the corresponding sum of the associated marginal risks and marginal costs of transaction and handling to exceed a positive lower bound. Moreover (cf. (41)), in the case where the demand price of the financial product in a currency and country as perceived by consumers at a demand market is high, we can expect that the demand for the financial product at the demand market to not exceed a positive bound.

We now establish additional qualitative properties both of the function F that enters the variational inequality problem (cf. (27)), as well as uniqueness of the equilibrium pattern. Since the proofs of Theorems 5 and 6 below are similar to the analogous proofs in Nagurney and Ke (2001) they are omitted here. Additional background on the properties established below can be found in the books by Nagurney and Siokos (1997) and Nagurney (1999).

Theorem 5: Monotonicity

Suppose that the risk function r^{il}; $i = 1, \ldots, I$; $l = 1, \ldots, L$, and r^j; $j = 1, \ldots, J$, are strictly convex and that the c^{il}_{jhm}, $c^{il}_{kh\hat{l}}$, c_j, $\hat{c}^{il}_{jhm}$, and $c^j_{kh\hat{l}m}$ functions are convex; the $\hat{c}^j_{kh\hat{l}m}$ and $\hat{c}^{il}_{kh\hat{l}}$ functions are monotone increasing, and the $d_{kh\hat{l}}$ functions are monotone decreasing functions, for all $i, l, j, h, k, \hat{l}, m$. Then the vector function F that enters the variational inequality (27) is monotone, that is,

$$\langle F(X') - F(X''), X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{42}$$

Monotonicity plays a role in the qualitative analysis of variational inequality problems similar to that played by convexity in the context of optimization problems. Under slightly stronger conditions than thos applied in Theorem 5, we now state the following result.

Theorem 6: Strict Monotonicity

Assume all the conditions of Theorem 5. In addition, suppose that one of the families of convex functions c^{il}_{jhm}; $i = 1, ..., I$; $l = 1, \ldots, L$; $j = 1, ..., J$; $h = 1, \ldots, H$; $m = 1, 2$; $c^{il}_{kh\hat{l}}$; $i = 1, ..., I$; $l = 1, \ldots, L$; $k = 1, ..., K$; $h = 1, \ldots, H$; $\hat{l} = 1, \ldots, L$; c_j; $j = 1, ..., J$; $\hat{c}^{il}_{jhm}$; $i = 1, \ldots, I$; $l = 1, \ldots, L$; $j = 1, \ldots, J$; $h = 1, \ldots, H$; $m = 1, 2$, and $c^j_{kh\hat{l}}$; $j = 1, \ldots, J$; $k = 1, \ldots, K$; $h = 1, \ldots, H$, and $\hat{l} = 1, \ldots, L$, is a family of strictly convex functions. Suppose also that $\hat{c}^j_{kh\hat{l}}$; $j = 1, ..., J$; $k = 1, ..., K$; $h = 1, \ldots, H$; $\hat{l} = 1, \ldots, L$; $\hat{c}^{il}_{kh\hat{l}}$; $i = 1, ..., I$; $l = 1, \ldots, L$; $k = 1, ..., K$; $h = 1, \ldots, H$; $\hat{l} = 1, \ldots, L$ and -$d_{kh\hat{l}}$; $k = 1, ..., K$; $h = 1, \ldots, H$; $\hat{l} = 1, \ldots, L$, are strictly monotone. Then, the vector function F that enters the variational inequality (27) is strictly monotone, with respect to (x^1, x^2, y, ρ_3), that is, for any two X', X'' with $(x^{1'}, x^{2'}, y^{'}, \rho_3') \neq (x^{1''}, x^{2''}, y^{''}, \rho_3'')$:

$$\langle F(X') - F(X''), X' - X'' \rangle > 0. \tag{43}$$

Theorem 7: Uniqueness

Assuming the conditions of Theorem 6, there must be a unique international financial flow pattern (x^{1*}, x^{2*}, y^*)*, and a unique demand price vector* ρ_3^* *satisfying the equilibrium conditions of the international financial network. In other words, if the variational inequality (27) admits a solution, then that is the only solution in* (x^1, x^2, y, ρ_3)*.*

Proof: Under the strict monotonicity result of Theorem 6, uniqueness follows from the standard variational inequality theory (cf. Kinderlehrer and Stampacchia (1980)) □

Theorem 7: Lipschitz Continuity

The function that enters the variational inequality problem (27) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq \mathcal{L}\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \textit{ where } \mathcal{L} > 0, \tag{44}$$

under the following conditions:

(i). the r^{il}*,* r^j*,* c^{il}_{jhm}*,* $c^{il}_{kh\hat{l}}$*,* c_j*,* $\hat{c}^{il}_{jhm}$*,* $c^j_{kh\hat{l}m}$ *functions have bounded second-order derivatives,* $\forall i, l, j, h, k, \hat{l}, m$*;*

(ii). the $\hat{c}^j_{khl}$*,* $\hat{c}^{il}_{kh\hat{l}}$*, and* $d_{kh\hat{l}}$ *functions have bounded first-order derivatives* $\forall i, l, j, k, h, \hat{l}$*.*

Proof: The result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (27). □

It is worth noting that the risk functions of the form (4) and (15) have bounded second-order derivatives.

Theorem 8: Existence of a Unique Trajectory

Assume the conditions of Theorem 7. Then, for any $X \in \mathcal{K}$*, there exists a unique solution* $X_0(t)$ *to the initial value problem (34).*

Proof: See Dupuis and Nagurney (1993) and Nagurney and Zhang (1996a). □

Note that unlike Theorems 3 and 4, this theorem is concerned with the existence of a unique dynamic trajectory and not the existence and uniqueness of an equilibrium pattern. We now, for completeness, provide a stability result(cf. Zhang and Nagurney (1995)). First, we recall the following:

Definition 2: Stability of a Systems

The system defined in (34) is stable if, for every X_0 and every equilibrium point X^*, the Euclidean distance $\|X^* - X_0(t)\|$ is a monotone nonincreasing function of time t.

We now provide a stability result.

Theorem 9: Stability of the International Financial Network

Assume the conditions of Theorem 5. Then the dynamical system (34) underlying the international financial network is stable.

Proof: Under the assumptions of Theorem 5, $F(X)$ is monotone, and, hence, the conclusion follows directly from Theorem 4.1 of Zhang and Nagurney (1995). $\square$

In the next Section, we propose the Euler method, which is a discrete-time algorithm and serves to approximate the dynamic trajectories of (34) and also yields the equilibrium international financial flow and price pattern.

5. The Discrete-Time Algorithm

In this Section, we consider the computation of solutions to variational inequality (26) (or (27)); equivalently, the stationary points of (34). The algorithm that we propose is the Euler-type method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993). The realization of the Euler method in the context of our model (for further details in the application of other financial network problems, see also Nagurney and Siokos (1997)) is as follows, where $\mathcal{T}$ denotes an iteration counter:

The Euler Method

Step 0: Initialization

Set $(x^{10}, x^{20}, y^0, \gamma^0, \rho_3^0) \in \mathcal{K}$. Let $\mathcal{T} = 1$ and set the sequence $\{a_{\mathcal{T}}\}$ so that $\sum_{\mathcal{T}=1}^{\infty} a_{\mathcal{T}}$, $a_{\mathcal{T}} > 0$, $a_{\mathcal{T}} \to 0$, as $\mathcal{T} \to \infty$ (such a sequence is required for convergence of the algorithm).

Step 1: Computation

Compute $(x^{1\mathcal{T}}, x^{2\mathcal{T}}, y^{\mathcal{T}}, \gamma^{\mathcal{T}}, \rho_3^{\mathcal{T}}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\Bigg[x_{jhm}^{il\mathcal{T}} + a_{\mathcal{T}}(\alpha^{il}\frac{\partial r^{il}(x^{il\mathcal{T}-1})}{\partial x_{jhm}^{il}} + \beta^j \frac{\partial r^j(x^{1\mathcal{T}-1}, y^{\mathcal{T}-1})}{\partial x_{jhm}^{il}}$$

$$+\frac{\partial c_{jhm}^{il}(x_{jhm}^{il\mathcal{T}-1})}{\partial x_{jhm}^{il}} + \frac{\partial c_j(x^{1\mathcal{T}-1})}{\partial x_{jhm}^{il}} + \frac{\partial \hat{c}_{jhm}^{il}(x_{jhm}^{il\mathcal{T}-1})}{\partial x_{jhm}^{il}} - \gamma_j^{\mathcal{T}-1}) - x_{jhm}^{il\mathcal{T}-1}\Bigg] \times \Big[x_{jhm}^{il} - x_{jhm}^{il\mathcal{T}}\Big]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\Bigg[x_{kh\hat{l}}^{il\mathcal{T}} + a_{\mathcal{T}}(\alpha^{il}\frac{\partial r^{il}(x^{il\mathcal{T}-1})}{\partial x_{kh\hat{l}}^{il}} + \frac{\partial c_{kh\hat{l}}^{il}(x_{kh\hat{l}}^{il\mathcal{T}-1})}{\partial x_{kh\hat{l}}^{il}}$$

$$+\hat{c}_{kh\hat{l}}^{il}(x^{2\mathcal{T}-1}) - \rho_{3kh\hat{l}}^{\mathcal{T}-1}) - x_{kh\hat{l}}^{il\mathcal{T}-1}\Bigg] \times \Big[x_{kh\hat{l}}^{il} - x_{kh\hat{l}}^{il\mathcal{T}}\Big]$$

$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\Bigg[y_{kh\hat{l}m}^{j\mathcal{T}} + a_{\mathcal{T}}(\beta^j\frac{\partial r^j(x^{1\mathcal{T}-1}, y^{\mathcal{T}-1})}{\partial y_{kh\hat{l}m}^{j}} + \frac{\partial c_{kh\hat{l}m}^{j}(y_{kh\hat{l}m}^{j\mathcal{T}-1})}{\partial y_{kh\hat{l}m}^{j}}$$

$$+\hat{c}_{kh\hat{l}m}^{j}(y^{\mathcal{T}-1}) - \rho_{3kh\hat{l}}^{\mathcal{T}-1} + \gamma_j^{\mathcal{T}-1}) - y_{kh\hat{l}m}^{j\mathcal{T}-1}\Bigg] \times \Big[y_{kh\hat{l}m}^{j} - y_{kh\hat{l}m}^{j\mathcal{T}}\Big]$$

$$+\sum_{j=1}^{J}\Bigg[\gamma_j^{\mathcal{T}} + a_{\mathcal{T}}(\sum_{m=1}^{2}\Bigg[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}x_{jhm}^{il\mathcal{T}-1} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}y_{kh\hat{l}m}^{j\mathcal{T}-1}\Bigg]) - \gamma_j^{\mathcal{T}-1}\Bigg] \times \Big[\gamma_j - \gamma_j^{\mathcal{T}}\Big]$$

$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\rho_{3kh\hat{l}}^{\mathcal{T}}+a_{\mathcal{T}}(\sum_{j=1}^{J}\sum_{m=1}^{2}y_{kh\hat{l}m}^{j\mathcal{T}-1}+\sum_{i=1}^{I}\sum_{l=1}^{L}x_{kh\hat{l}}^{il\mathcal{T}-1}-d_{kh\hat{l}}(\rho_3^{\mathcal{T}-1}))-\rho_{3kh\hat{l}}^{\mathcal{T}-1}\right]\times\left[\rho_{3kh\hat{l}}-\rho_{3kh\hat{l}}^{\mathcal{T}}\right]\geq 0,$$

$$\forall(x^1,x^2,y,\gamma,\rho_3)\in\mathcal{K}. \quad (45)$$

Step 2: Convergence Verification

If $|x_{jhm}^{il\mathcal{T}}-x_{jhm}^{il\mathcal{T}-1}|\leq\epsilon$,$|x_{kh\hat{l}}^{il\mathcal{T}}-x_{kh\hat{l}}^{il\mathcal{T}-1}|\leq\epsilon$, $|y_{kh\hat{l}m}^{j\mathcal{T}}-y_{kh\hat{l}m}^{j\mathcal{T}-1}|\leq\epsilon$, $|\gamma_j^{\mathcal{T}}-\gamma_j^{\mathcal{T}-1}|\leq\epsilon$, $|\rho_{3kh\hat{l}}^{\mathcal{T}}-\rho_{3kh\hat{l}}^{\mathcal{T}-1}|\leq\epsilon$, for all $i=1,\cdots,I$; $l=1,\ldots,L$; $\hat{l};\hat{l}=1,\ldots,L$; $m=1,2$; $j=1,\cdots,J$; $h=1,\ldots,H$; $k=1,\cdots,K$, with $\epsilon>0$, a pre-specified tolerance, then stop; otherwise, set $\mathcal{T}:=\mathcal{T}+1$, and go to Step 1.

Variational inequality subproblems (45) can be solved explicitly and in closed form since the induced subproblems are actually quadratic programming problems and the feasible set is a Cartesian product consisting of the the product of $\mathcal{K}^1$ and $\mathcal{K}^2$. The former has a simple network structure, whereas the latter consists of the cross product of the nonnegative orthants: R_+^{2ILJH}, R_+^J, and R_+^{KHL}, and corresponding to the variables y, γ, and ρ_3, respectively. In fact, the subproblems in (45) corresponding to the x variables can be solved using exact equilibration (cf. Dafermos and Sparrow (1969) and Nagurney (1999)), whereas the remainder of the variables in (45) can be obtained by explicit formulae.

We now, for completeness, and also to illustrate the simplicity of the proposed computational procedure in the context of the international financial network model, state the explicit formulae for the computation of the $y^{\mathcal{T}}$, the $\gamma^{\mathcal{T}}$, and the $\rho_3^{\mathcal{T}}$ (cf. (45)).

Computation of the Financial Products from the Intermediaries

In particular, compute, at iteration $\mathcal{T}$, the $y_{kh\hat{l}m}^{j\mathcal{T}}$s, according to:

$$y_{kh\hat{l}m}^{j\mathcal{T}}=\max\{0,y_{kh\hat{l}m}^{j\mathcal{T}-1}-a_{\mathcal{T}}(\beta^j\frac{\partial r^j(x^{1\mathcal{T}-1},y^{\mathcal{T}-1})}{\partial y_{kh\hat{l}m}^j}+\frac{\partial c_{kh\hat{l}m}^j(y_{kh\hat{l}m}^{j\mathcal{T}-1})}{\partial y_{kh\hat{l}m}^j}+\hat{c}_{kh\hat{l}m}^j(y^{\mathcal{T}-1})-\rho_{3kh\hat{l}}^{\mathcal{T}-1}+\gamma_j^{\mathcal{T}-1})\},$$

$$\forall j,k,h,\hat{l},m. \quad (46)$$

Computation of the Prices

At iteration $\mathcal{T}$, compute the $\gamma_j^{\mathcal{T}}$s according to:

$$\gamma_j^{\mathcal{T}} = \max\{0, \gamma_j^{\mathcal{T}-1} - a_{\mathcal{T}}(\sum_{m=1}^{2}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H} x_{jhm}^{il\mathcal{T}-1} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} y_{kh\hat{l}m}^{j\mathcal{T}-1}\right])\}, \quad \forall j, \tag{47}$$

whereas the $\rho_{3kh\hat{l}}^{\mathcal{T}}$s are computed explicitly and in closed form according to:

$$\rho_{3kh\hat{l}}^{\mathcal{T}} = \max\{0, \rho_{3kh\hat{l}}^{\mathcal{T}-1} - a_{\mathcal{T}}(\sum_{j=1}^{J}\sum_{m=1}^{2} y_{kh\hat{l}m}^{j\mathcal{T}-1} + \sum_{i=1}^{I}\sum_{l=1}^{L} x_{kh\hat{l}}^{il\mathcal{T}-1} - d_{kh\hat{l}}(\rho_3^{\mathcal{T}-1}))\}, \quad \forall k, h, \hat{l}. \tag{48}$$

In the next Section, we apply the Euler method to solve several international financial network examples. Convergence results for this algorithm may be found in Dupuis and Nagurney (1993) and for a variety of applications in Nagurney and Zhang (1996a).

6. Numerical Examples

In this Section, we apply the Euler method to several international financial network examples. The Euler method was implemented in FORTRAN and the computer system used was a Sun system located at the University of Massachusetts at Amherst. For the solution of the induced network subproblems in the (x^1, x^2) variables we utilized the exact equilibration algorithm (see Dafermos and Sparrow (1969), Nagurney (1999), and the references therein). The other variables were determined explicitly and in closed form as described in the preceding section.

The convergence criterion used was that the absolute value of the flows and prices between two successive iterations differed by no more than 10^{-4}. The sequence $\{a_{\mathcal{T}}\}$ used for all the examples was: $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots$ in the algorithm.

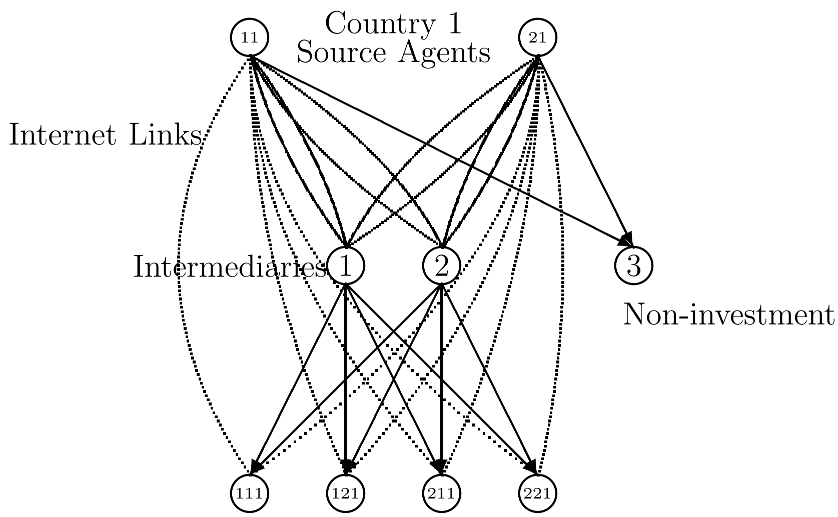
We assumed in all the examples that the risk functions were of the form (4) and (15), that is, that risk was represented through variance-covariance matrices for both the source agents in the countries and for the intermediaries. We initialized the Euler method as follows: We set $x^{il}_{jh1} = \frac{S^{il}}{JH}$ for each source agent i and country l and for all j and h. All the other variables were initialized to zero.

Example 1

The first numerical example consisted of one country, two source agents, two currencies, two intermediaries, and two financial products. Hence, $L = 1$, $I = 2$, $H = 2$, $J = 2$, and $K = 2$. The international financial network for the first example is depicted in Figure 2. In this example the electronic transactions were between the source agents and the demand markets.

The data for the first example were constructed for easy interpretation purposes. The financial holdings of the two source agents were: $S^{11} = 20$ and $S^{21} = 20$. The variance-covariance matrices Q^{il} and Q^j were equal to the identity matrices (appropriately dimensioned) for all source agents and all intermediaries, respectively. Note that since only physical transactions are allowed (except for as stated above), we have that $m = 1$.

The transaction cost functions faced by the source agents associated with transacting



Product/Currency/Country Combination

Figure 2: International financial network for Example 1

with the intermediaries (cf. (1a)) were given by:

$$c^{il}_{jhm}(x^{il}_{jhm}) = .5(x^{il}_{jhm})^2 + 3.5x^{il}_{jhm}; \quad i = 1,2; l = 1; j = 1,2; h = 1,2; m = 1.$$

The analogous transaction costs but associated with the electronic transactions between source agents and demand markets (cf. (1b)) were given by:

$$c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}) = .5(x^{il}_{kh\hat{l}})^2 + x_{kh\hat{l}}; \quad \forall i, l, \hat{l}, k, h.$$

The handling costs of the intermediaries, in turn (see (10)), were given by:

$$c_j(x^1) = .5(\sum_{i=1}^{2}\sum_{h=1}^{2} x^{i1}_{jh1})^2; \quad j = 1,2.$$

The transaction costs of the intermediaries associated with transacting with the source agents were (cf. (11)) given by:

$$\hat{c}^{il}_{jhm}(x^{il}_{jhm}) = 1.5{x^{il}_{jhm}}^2 + 3x^{il}_{jhm};\ i = 1,2; l = 1; j = 1,2; h = 1,2; m = 1.$$

The transaction costs, in turn, associated with the electronic transactions at the demand markets (from the perspective of the consumers (cf. (20b)) were given by:

$$\hat{c}^{il}_{kh\hat{l}}(x^2) = .1x^{il}_{kh\hat{l}} + 1, \quad \forall i, l, \hat{l}, k, h.$$

The demand functions at the demand markets (refer to (3)) were:

$$d_{111}(\rho_3) = -2\rho_{3111} - 1.5\rho_{3121} + 1000, \quad d_{121}(\rho_3) = -2\rho_{3121} - 1.5\rho_{3111} + 1000,$$

$$d_{211}(\rho_3) = -2\rho_{3211} - 1.5\rho_{3221} + 1000, \quad d_{221}(\rho_3) = -2\rho_{3221} - 1.5\rho_{3211} + 1000.$$

and the transaction costs between the intermediaries and the consumers at the demand markets (see (20a)) were given by:

$$\hat{c}^{j}_{kh\hat{l}m}(y) = y^{j}_{kh\hat{l}m} + 5; \;\; k = 1, 2; h = 1, 2; \hat{l} = 1; m = 1.$$

In Example 1 all of the weights associated with the risk functions, that is, the α^{il} and the β^j were set equal to 1 for all i, l, and j. This means that the source agents as well as the intermediaries weight the criterion of risk minimization equally to that of net revenue maximization.

The Euler method converged in 1,087 iterations and yielded the following equilibrium financial flow pattern:

$$x^{1*} := x^{11*}_{111} = x^{11*}_{121} = x^{11*}_{211} = x^{11*}_{221} = x^{211}_{11*} = x^{22*}_{211} = x^{21*}_{211} = x^{21*}_{221} = .372;$$

$$x^{2*} := x^{11*}_{111} = x^{11*}_{121} = x^{11*}_{211} = x^{11*}_{221} = x^{21*}_{111} = x^{21*}_{121} = x^{21*}_{211} = x^{21*}_{221} = 4.627;$$

$$y^{*} := y^{1*}_{1111} = y^{1*}_{1211} = y^{1*}_{2111} = y^{1*}_{2211} = y^{2*}_{1111} = y^{2*}_{1211} = y^{2*}_{2111} = y^{2*}_{2211} = .372.$$

Both source agents allocated the entirety of their funds to the instrument in the two currencies; thus, there was no non-investment.

The vector γ^* had components: $\gamma^*_1 = \gamma^*_2 = 276.742$, and the computed demand prices at the demand markets were: $\rho^*_{3111} = \rho^*_{3121} = \rho^*_{3211} = \rho^*_{3221} = 282.858$.

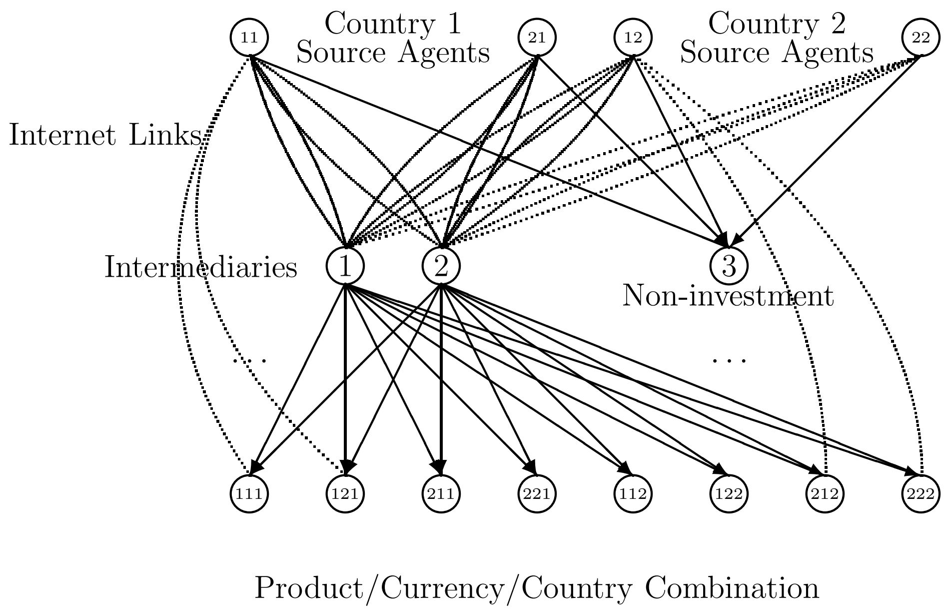


Figure 3: International financial network for Example 2

We also computed (as discussed following (27)) the equilibrium prices associated with the top tier of nodes in the international financial network and the equilibrium vector ρ_1^* had all of its components equal to 270.386.

Note that due to the lower transaction costs associated with electronic transactions directly between the source agents and the demand markets a sizeable portion of the financial funds were transacted in this manner.

Example 2

For Example 2, the international financial network was as given in Figure 3. The example consisted of two countries with two source agents in each country; two currencies, two intermediaries, and two financial products. Hence, $L = 2$, $I = 2$, $H = 2$, $J = 2$, and $K = 2$.

The data for Example 2 was constructed for easy interpretation purposes and to create a baseline from which additional simulations could be conducted. In fact, we essentially "replicated" the data for the first country as it appeared in Example 1 in order to construct the data for the second country.

Specifically, the financial holdings of the source agents were: $S^{11} = 20$, $S^{21} = 20$, $S^{12} = 20$, and $S^{22} = 20$. The variance-covariance matrices Q^{il} and Q^{j} were equal to the identity matrices (appropriately dimensioned) for all source agents in each country and for all intermediaries, respectively.

The transaction cost functions faced by the source agents associated with transacting with the intermediaries were given by:

$$c^{il}_{jhm}(x^{il}_{jhm}) = .5(x^{il}_{jhm})^2 + 3.5x^{il}_{jhm};\ \ i = 1,2; l = 1,2; j = 1,2; h = 1,2; m = 1.$$

The handling costs of the intermediaries (since the number of intermediaries is still equal to two) remained as in Example 1, that is, they were given by:

$$c_j(x^1) = .5(\sum_{i=1}^{2}\sum_{h=1}^{2} x^{i1}_{jh1})^2;\ j = 1,2.$$

The transaction costs of the intermediaries associated with transacting with the source agents in the two countries were given by:

$$\hat{c}^{il}_{jhm}(x^{il}_{jhm}) = 1.5(x^{il}_{jhm})^2 + 3x^{il}_{jhm};\ i = 1,2; l = 1,2; j = 1,2; h = 1,2; m = 1.$$

The demand functions at the demand markets were:

$$d_{111}(\rho_3) = -2\rho_{3111} - 1.5\rho_{3121} + 1000, \quad d_{121}(\rho_3) = -2\rho_{3121} - 1.5\rho_{3111} + 1000,$$

$$d_{211}(\rho_3) = -2\rho_{3211} - 1.5\rho_{3221} + 1000, \quad d_{221}(\rho_3) = -2\rho_{3221} - 1.5\rho_{3211} + 1000,$$

$$d_{112}(\rho_3) = -2\rho_{3112} - 1.5\rho_{3122} + 1000, \quad d_{122}(\rho_3) = -2\rho_{3122} - 1.5\rho_{3112} + 1000,$$

$$d_{212}(\rho_3) = -2\rho_{3212} - 1.5\rho_{3222} + 1000, \quad d_{222}(\rho_3) = -2\rho_{3222} - 1.5\rho_{3212} + 1000,$$

and the transaction costs between the intermediaries and the consumers at the demand markets were given by:

$$\hat{c}^{j}_{kh\hat{l}}(y) = y^{j}_{kh\hat{l}} + 5;\ \ j = 1,2; k = 1,2; h = 1,2; \hat{l} = 1,2; m = 1.$$

The data for the electronic links were as in Example 1 and were replicated for the other source agents.

The variance-covariance matrices were redimensioned and were equal to the identity matrices. The weights associated with the risk functions were set equal to 1 for all the source agents and intermediaries.

The Euler method converged in 1,826 iterations and yielded the following equilibrium international financial flow pattern: only the electronic links had positive flows with all other flows being identically equal to 0.000. In particular, the financial holdings of the source agents in the different countries were equally allocated via electronic transactions directly to the demand markets with $x^{il*}_{kh\hat{l}} = 2.5000$ for all $i, l, k, h, \hat{l}$.

The vector γ^* had components: $\gamma^*_1 = \gamma^*_2 = 279.6194$, and the computed demand prices at the demand markets were: $\rho^*_{3kh\hat{l}} = 282.8578$, $\forall k, h, \hat{l}$. In this example, all the financial transactions were conducted electronically.

These examples, although stylized, have been presented to show both the model and the computational procedure. Obviously, different input data and dimensions of the problems solved will affect the equilibrium financial flow and price patterns. One now has a powerful tool with which to explore the effects of perturbations to the data as well as the effects of changes in the number of source agents, countries, currencies, and/or products, as well as the effects of the introduction of electronic transactions.

Note that the reason the Euler method requires many iterations for convergence is that it also provides a time discretization of the financial flow and price trajectories.

7. Summary and Conclusions

In this paper, we developed a framework for the modeling, analysis, and computation of solutions to international financial problems with intermediaries with multicriteria decision-makers in the presence of electronic transactions. We proposed an international financial network model consisting of three tiers of decision-makers: the source agents, the financial intermediaries, and the consumers associated with the demand markets for distinct financial products in distinct currencies and countries. We modeled the behavior of the decision-makers, with the first two tiers being concerned both with net revenue maximization and with risk minimization (with appropriate weights assigned to the latter), derived the optimality conditions as well as the governing equilibrium conditions which reflect the (possible) competition among decision-makers at the same tier of nodes but cooperation between tiers of nodes. The framework allows for the handling of as many countries, as many source agents in each country, as many currencies in which the financial products can be obtained, and as many financial intermediaries, as mandated by the specific application. The formulation of the equilibrium conditions was shown to be equivalent to a finite-dimensional variational inequality problem.

We then proposed a dynamic adjustment process which provides the disequilibrium dynamics of the international financial flows and prices and formulated it as a projected dynamical system. We further demonstrated that the set of stationary points of the projected dynamical system coincides with the set of solutions of the variational inequality formulation of the international financial network equilibrium.

We provided a variety of qualitative properties of the equilibrium financial flow and price pattern, and also established, under reasonable conditions, the stability of the international financial network.

Finally, we proposed a time-discretization of the continuous time dynamic adjustment process in the form of an Euler-type method which allows for closed form expressions of the prices and certain financial flows at each time period. The Euler method was then applied to solve international financial network examples.

This framework generalizes the recent work of Nagurney and Ke (2003) to the international arena and that of Nagurney and Cruz (2002) to include dynamics and multicriteria decision-making.

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