**Transportation Network Equilibrium Reformulations**
**of**
**Electric Power Supply Chain Networks**
**with**
**Computations**

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## Abstract

In this paper we demonstrate rigorously that electric power supply chain network equilibrium problems can be reformulated and solved as transportation network equilibrium problems. This connection was originally hypothesized in the classic book of Beckmann, McGuire, and Winsten (1956) in a chapter dedicated to "some unsolved problems." We consider two distinct electric power supply chain network models with known demand functions and with known demand market price functions, respectively, and prove that the variational inequality formulations of the governing equilibrium conditions coincide with the corresponding variational inequalities of transportation network equilibrium problems over appropriately constructed supernetworks. This equivalence provides new interpretations of electric power supply chain network equilibria in terms of paths and path flows and also allows for the transfer of methodological tools developed for transportation network equilibrium problems to the electric power network application domain. Numerical examples illustrate the potential of the theoretical results obtained in this paper.

*Keywords*: Electric power generation and distribution networks; Supply chain networks; Transportation network equilibrium; User-Optimization; Supernetworks; Variational Inequalities

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## 1. Introduction

Beckmann, McGuire, and Winsten in their 1956 classic book, **Studies in the Economics of Transportation**, laid down the mathematical foundations for the rigorous modeling and analysis of congested transportation networks, operating under two distinct behavioral principles due to Wardrop (1952), which are now known as user-optimal and system-optimized networks. In particular, among other innovations, they proposed an elastic travel demand model with separable user link cost and travel disutility functions, and proved that the transportation network equilibrium conditions, corresponding to user-optimization, coincided with the Kuhn-Tucker conditions of an appropriately constructed optimization problem. Beckmann, McGuire, and Winsten (1956) recognized the generality of the concepts introduced in their book and identified other potential applications of network equilibrium. Boyce, Mahmassani, and Nagurney (2005) discuss the impact of this book not only on transportation science but also on computer science, a subject, which did not even exist at the time of the book's publication.

The fifth chapter of the Beckmann, McGuire, and Winsten (1956) book described some "unsolved problems" including a single commodity network equilibrium problem that the authors intuited could be generalized to capture electric power generation and distribution networks. In this paper, we take up that challenge of establishing the relationship and application of transportation network equilibrium models to electric power generation and distribution networks or supply chains. As noted in that classic book on page 106, "The unsolved problems concern the application of this model to particular cases." "In particular, the problem of generation and distribution of electric energy in a network comes to mind." Central to the models developed by Beckmann, McGuire, and Winsten (1956) was the mathematical description of the underlying economic behavior of the decision-makers on the transportation networks.

In particular, in this paper, we demonstrate how competitive electric power supply chain networks, consisting of power generators, competitive retailers (or suppliers), transmission providers, as well as consumers at the demand markets, can be reformulated and solved as transportation network equilibrium problems. More precisely, we prove that the equilibrium of the electricity transactions between/among the non-cooperative decision-makers in the competitive electric power supply chain network is isomorphic to the user-optimal equilibrium of a properly configured transportation network. This rigorous supernetwork equivalence, which requires that a path flow formulation of the electric power supply chain network equilibrium be established, exists because of the following reasons: On one hand, the economic competition among the non-cooperative decision-makers in the wholesale and retail markets leads to Nash equilibria in the electricity supply chain network, but with the transactions between tiers of decision-makers requiring cooperation, and, hence, a path concept. On the other hand, the transportation network with non-cooperative road users also results in a Nash equilibrium (cf. Dafermos and Sparrow (1969) and Nagurney (1993) and the references therein) so that no decision-maker can be better off by unilaterally altering its decision. We show that the respective equilibria share the same mathematical representation in the context of finite-dimensional variational inequalities and over an appropriately constructed abstract network or supernetwork (cf. Nagurney and Dong (2002)).

The electric power industry in the United States, as well as abroad, is undergoing a profound transformation in the way it delivers electricity to millions of residential and commercial customers. Electricity production was once dominated by vertically integrated investor-owned utilities who owned many of the generation capacity, transmission, and distribution facilities. However, electricity deregulation has opened up this monopoly system to competition, and has made the electric power industry today characterized by many new companies that produce, market wholesale and retail electric power. By February 1, 2005, in the US, 1200 companies were registered to sell wholesale power, and more than 400 retail companies were certified to provide electricity retail service to the end-customers (statistics available at http://www.eia.doe.gov).

The emerging competitive markets have fundamentally changed the electricity trading patterns as well as the structure of electric power supply chains. These changes have stimulated research activity in the area of electric power supply systems modeling and analysis during the past decade. Smeers (1997) reviewed and discussed a wide range of equilibrium models of electricity and gas markets with various market power assumptions. Some models have been developed to study the more decentralized electric power markets (see, e.g., Schweppe et al. (1988), Hogan (1992), Chao and Peck (1996), and Wu et al. (1996)). Some researchers have suggested different variations of the models depending on the electric power market organizational structure (see, for example, Hobbs (2001) and Metzler, Hobbs, and Pang (2003)). Hogan (1997) proposed a market power model with strategic interactions in an electric power transmission network. Chen et al. (2004) constructed a Stackelberg game to study the possibility that the leading generator in the electric power market influenced both the electricity and emission allowances markets to its advantage. Baldick, Grant, and Kahn (2004) presented linear supply function equilibrium (SFE) models which generalized the linear SFE model initially applied by Green and Newbery (1992) and Green (1996). Several simulation models have also been proposed to model the interaction of competing generation firms who strategically make pricing decisions (see Kahn (1998) and Hobbs, Metzler, and Pang (2000)). Day, Hobbs, and Pang (2002) simulated the exercising of market power on linearized dc networks based on a flexible representation of interactions of competing generating firms.

Nagurney and Matsypura (2004) proposed a novel electric power supply chain network model which provided a competitive supply chain network perspective for electric power generation, supply, transmission, and consumption and captured the decentralized decision-making behaviors of the various players involved. The models presented in this paper and in Nagurney and Matsypura (2004) differ from electric power network models in the literature in the following way. First, our models consider the entire electric power supply chain network that includes both the competitive wholesale market and the competitive retail (supply) market, and capture the economic behaviors of all the decision-makers in the electric power supply chain network including the generators, the competitive retailers, the transmission service providers, and the end-consumers. Second, our models are focused on the the economic decision-making processes of the players and the electricity transactions from generation through ultimate consumption, along with the associated prices. For example, the competitive retailers or suppliers may not actually operate any facility and physically possess electricity at any stage of the supply chain. However, they play very important roles in both the competitive wholesale market and the competitive retail market, and, therefore, they are explicitly represented as decision-makers in our models.

In this paper, we also give two alternative model formulations, in the case of known demand functions and known demand market price functions. This allows for greater modeling flexibility and also yields a complete mapping between electric power supply chain networks and elastic demand transportation networks. Moreover, we provide an entirely new economic interpretation of the equilibrium conditions for competitive electric power supply chain networks in terms of paths and path flows revealed through the transportation network equilibrium reformulations. This equivalence allows one to transfer the methodological tools originally proposed for transportation network equilibrium problems to electric power supply chain networks.

This paper is organized as follows. In Section 2, we present the electric power supply chain network models in which the behavior of the various economic decision-makers associated with the nodes of the network is made explicit. We also derive the governing variational inequality formulations and contrast them with the formulation obtained in the case of known demand functions by Nagurney and Matsypura (2004). Additional background on electric power systems can be found in the book by Casazza and Delea (2003) and in the edited volumes by Zaccour (1998) and Singh (1999). Of course, we also recognize the work of Duffin (1947) that utilized optimization concepts to model electrical circuits following Kirchoff's laws. The focus of this paper, however, is not on the operational level of electric power networks but on the strategic supply chain level.

In Section 3, we recall the elastic demand transportation network equilibrium models, along with their variational inequality formulations, due to Dafermos (1982) and Dafermos and Nagurney (1984a). Note that Dafermos (1980) recognized that the transportation network equilibrium conditions as stated by Smith (1979) corresponded to a finite-dimensional variational inequality problem. Of course, in the special case in which the user link cost functions (and the travel disutility functions) admit symmetric Jacobians, as was the case in the classical models of Beckmann, McGuire, and Winsten (1956), the equilibrium conditions coincide with the Kuhn-Tucker conditions of an appropriately constructed optimization problem (see also, e.g., Nagurney (1993), Patriksson (1994)).

In Section 4, we then show how the electric power supply chain network models of Section 2 can be reformulated, through a supernetwork equivalence, as transportation network equilibrium models with elastic demands as described in Section 3. This equivalence allows us to transfer the theory of transportation network equilibrium modeling, analysis, and computations to the formulation and analysis of electric power supply chains. In Section 5, we then explore the applicability of the theoretical results established in Section 4 by utilizing algorithms developed for transportation network equilibrium problems in order to determine the equilibrium power flow and price patterns in several numerical electric power supply chain network examples.

The results in this paper demonstrate, as originally speculated in Beckmann, McGuire, and Winsten (1956), that electric power generation and distribution network problems can be reformulated as transportation network equilibrium problems. We summarize the results obtained in this paper in Section 6, in which we also present suggestions for future research.

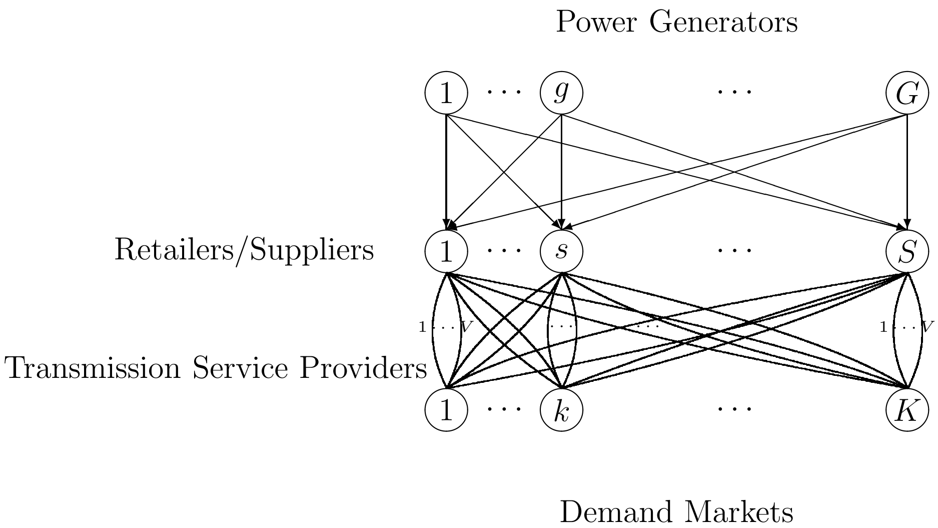


Figure 1: The Electric Power Supply Chain Network

## 2. The Electric Power Supply Chain Network Models

In this Section, we use, as the starting point of our electric power supply chain network model development, the electric power supply chain network equilibrium model proposed in Nagurney and Matsypura (2004). Here, however, we will assume that, since electric power cannot be stored, the electric power available at each retailer/supplier is equal to the electric power transmitted. In addition, we provide two alternative formulations depending upon whether the demand functions or the demand price functions are given. We consider $G$ power generators, $S$ competitive retailers or suppliers, $V$ transmission service providers, and $K$ consumer markets, as depicted in Figure 1. The majority of the needed notation is given in Table 1. An equilibrium solution is denoted by "*". All vectors are assumed to be column vectors, except where noted otherwise.

Table 1: Notation for the Electric Power Supply Chain Network Models

| **Notation** | **Definition** |
|---|---|
| $q$ | vector of the power generators' electric power outputs with components: $q_1, \ldots, q_G$ |
| $Q^1$ | $GS$-dimensional vector of electric power flows between power generators and retailers/suppliers with component $gs$ denoted by $q_{gs}$ |
| $Q^2$ | $SKV$-dimensional vector of power flows between retailers/suppliers and demand markets with component $svk$ denoted by $q^v_{sk}$ and denoting the flow between retailer/supplier $s$ and demand market $k$ via transmission provider $v$ |
| $\rho_1$ | $GS$-dimensional vector of prices charged by the generators in transacting with the retailers/suppliers with component $gs$ denoted by $\rho_{1gs}$ |
| $\rho_2$ | $SKV$-dimensional vector of prices charged by the retailers/suppliers in transacting with the demand markets with component $skv$ denoted by $\rho^v_{2sk}$ |
| $\rho_3$ | $K$-dimensional vector of demand market prices with component $k$ denoted by $\rho_{3k}$ |
| $d$ | $K$-dimensional vector of satisfied market demand with component $k$ denoted by $d_k$ |
| $f_g(q)$ $\equiv f_g(Q^1)$ | power generating cost function of power generator $g$ with marginal power generating cost with respect to $q_g$ denoted by $\frac{\partial f_g}{\partial q_g}$ and the marginal power generating cost with respect to $q_{sg}$ denoted by $\frac{\partial f_g(Q^1)}{\partial q_{gs}}$ |
| $c_{gs}(Q^1)$ | transaction cost incurred by power generator $g$ in transacting with retailer/supplier s with marginal transaction cost denoted by $\frac{\partial c_{gs}(Q^1)}{\partial q_{gs}}$ |
| $h$ | $S$-dimensional vector of the retailers' supplies of the electric power with component $s$ denoted by $h_s$, with $h_s \equiv \sum_{g=1}^{G} q_{gs}$ |
| $c_s(h)$ $\equiv c_s(Q^1)$ | operating cost of retailer/supplier $s$ with marginal operating cost with respect to $h_s$ denoted by $\frac{\partial c_s}{\partial h_s}$ and the marginal operating cost with respect to $q_{gs}$ denoted by $\frac{\partial c_j(Q^1)}{\partial q_{gs}}$ |
| $c^v_{sk}(Q^2)$ | transaction cost incurred by power retailer/supplier $s$ in transacting with demand market $k$ via transmission provider $v$ with marginal transaction cost with respect to $q^v_{sk}$ denoted by $\frac{\partial c^v_{sk}(Q^2)}{\partial q^v_{sk}}$ |
| $\hat{c}_{gs}(Q^1)$ | transaction cost incurred by retailer/supplier s in transacting with power generator $g$ with marginal transaction cost denoted by $\frac{\partial \hat{c}_{gs}(Q^1)}{\partial q_{gs}}$ |
| $\hat{c}^v_{sk}(Q^2)$ | unit transaction cost incurred by consumers at demand market $k$ in transacting with competitive retailer $s$ via transmission provider $v$ |
| $d_k(\rho_3)$ | demand function at demand market $k$ |
| $\rho_{3k}(d)$ | inverse demand (demand market price) function at demand market $k$ |
| $R^X_+$ | X-dimensional cone with non-negative coordinates |

Note that, in Table, 1 $d_k$ and $d_k(\rho_3)$ are entirely different notations: $d_k$ is the value of the satisfied market demand at consumption market $k$ while $d_k(\rho_3)$ represents the demand function at market $k$. Similarly, $\rho_{3k}$ is the electricity price at demand market $k$ while $\rho_{3k}(d)$ is the price function (inverse demand function) at demand market $k$.

The top-tiered nodes in the electric power supply chain network in Figure 1 represent the $G$ electric power generators, who are the decision-makers who own and operate the electric power generating facilities or power plants. They produce electric power and sell to the power retailers/suppliers in the second tier. We assume that each electric power generator seeks to determine the optimal production and allocation of the electric power in order to maximize his own profit.

Electric power retailers/suppliers, which are represented by the second-tiered nodes in Figure 1, function as intermediaries. They purchase electric power from the power generators and sell to the consumers at the different demand markets. We assume that the retailers compete with one another in a noncooperative manner. However, the retailers do not physically possess electric power at any stage of the supplying process; they only hold and trade the rights for the electric power. Therefore, the link connecting a power generator and retailer/supplier pair represents the decision-making connectivity and the transaction of the right of electric power between the two entities.

The bottom-tiered nodes in Figure 1 represent the demand markets, which can be distinguished from one another by their geographic locations or the type of associated consumers such as whether they correspond, for example, to businesses or to households.

A transmission service is necessary for the physical delivery of electric power from the power generators to the points of consumption. The transmission service providers are the entities who own and operate the electric power transmission and distribution systems, and distribute electric power from power generators to the consumption markets. However, since these transmission service providers do not make decisions such as to where or from whom the electric power will be delivered, they are not explicitly represented by nodes in this network model. We, instead, following Nagurney and Matsypura (2004), model them as different modes of transaction corresponding to the parallel links connecting a given retailer node to a given demand market node in Figure 1. Hence, an implicit assumption is that the suppliers need to cover the direct cost and decide which transmission service providers should be used and how much electric power should be delivered.

Now, for completeness and easy reference, we describe the behavior of the electric power generators, the retailers/suppliers, and the consumers at the demand markets. We then state the equilibrium conditions of the electric power network and provide the variational inequality formulations. We, subsequently, contrast the derived variational inequalities with the one obtained by Nagurney and Matsypura (2004).

The feasible sets underlying the decision-makers' optimization problems as well as those are defined when they first appear in the paper. The definitions of all the feasible sets are also collected, for convenience of the reader, in Table 2 in Appendix 1.

### The Behavior of Power Generators and their Optimality Conditions

Since for each individual power generator the total amount of electric power sold cannot exceed the total production of electric power, the following conservation of flow equations must hold for each power generator:

$$\sum_{s=1}^{S} q_{gs} = q_g, \quad g = 1, \ldots, G. \tag{1}$$

Let $\rho^*_{1gs}$ denote the unit price charged by power generator $g$ for the transaction with retailer $s$. $\rho^*_{1gs}$ is an endogenous variable and can be determined once the complete network equilibrium model is solved. Since we have assumed that each individual power generator is a profit-maximizer, the optimization problem of power generator $g$ can be expressed as follows:

$$\text{Maximize} \quad \sum_{s=1}^{S} \rho^*_{1gs} q_{gs} - f_g(Q^1) - \sum_{s=1}^{S} c_{gs}(Q^1) \tag{2}$$

subject to:

$$q_{gs} \geq 0, \qquad s = 1, \ldots, S. \tag{3}$$

We assume that the generating cost and the transaction cost functions for each power generator are continuously differentiable and convex, and that the power generators compete in a noncooperative manner in the sense of Cournot (1838) and Nash (1950, 1951). The optimality conditions for all power generators simultaneously, under the above assumptions (see also Gabay and Moulin (1980), Bazaraa, Sherali, and Shetty (1993), Bertsekas and Tsitsiklis (1989), and Nagurney (1993)), coincide with the solution of the following variational inequality: determine $Q^{1*} \in R_+^{GS}$ satisfying

$$\sum_{g=1}^{G} \sum_{s=1}^{S} \left[ \frac{\partial f_g(Q^{1*})}{\partial q_{gs}} + \frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}} - \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*] \geq 0, \quad \forall Q^1 \in R_+^{GS}. \tag{4}$$

As defined in Table 1, the power generating cost $f_g$ is a function of the total electric power productions, that is:

$$f_g(q) \equiv f_g(Q^1). \tag{5}$$

Hence, the marginal power generating cost with respect to $q_g$ is equal to the marginal generating cost with respect to $q_{gs}$:

$$\frac{\partial f_g(q)}{\partial q_g} \equiv \frac{\partial f_g(Q^1)}{\partial q_{sg}}, \quad g = 1, \ldots, G. \tag{6}$$

Using (1) and (6), we can transform (4) into the following equivalent variational inequality: determine $(q^*, Q^{1*}) \in \mathcal{K}^1$ satisfying

$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{g=1}^{G} \sum_{s=1}^{S} \left[ \frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}} - \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*] \geq 0, \quad \forall (q, Q^1) \in \mathcal{K}^1, \tag{7}$$

where $\mathcal{K}^1 \equiv \{(q, Q^1) | (q, Q^1) \in R_+^{G+GS}$ and (1) holds$\}$.

**The Behavior of Power Retailers/Suppliers and their Optimality Conditions**

The retailers or suppliers, such as the power marketers, traders, and brokers, in turn, are involved in transactions both with the power generators and with the consumers at demand markets through the transmission service providers.

Since electric power cannot be stored, it is reasonable to assume that the total amount of electricity sold by a retailer/supplier is equal to the total electric power that he purchased from the generators. This assumption can be expressed as the following conservation of flow equations:

$$\sum_{k=1}^{K} \sum_{v=1}^{V} q_{sk}^v = \sum_{g=1}^{G} q_{gs}, \quad s = 1, \ldots, S. \tag{8}$$

In Nagurney and Matsypura (2004), in contrast, it was assumed that (8) was an inequality.

Let $\rho_{2sk}^{v*}$ denote the price charged by retailer $s$ to demand market $k$ via transmission service provider $v$. This price is determined endogenously in the model once the entire network equilibrium problem is solved. As noted above, it is assumed that each retailer/supplier seeks to maximize his own profit. Hence the optimization problem faced by retailer $s$ may be expressed as follows:

$$\text{Maximize} \quad \sum_{k=1}^{K}\sum_{v=1}^{V}\rho_{2sk}^{v*}q_{sk}^{v} - c_s(Q^1) - \sum_{g=1}^{G}\rho_{1gs}^{*}q_{gs} - \sum_{g=1}^{G}\hat{c}_{gs}(Q^1) - \sum_{k=1}^{K}\sum_{v=1}^{V}c_{sk}^{v}(Q^2) \tag{9}$$

subject to:

$$\sum_{k=1}^{K}\sum_{v=1}^{V}q_{sk}^{v} = \sum_{g=1}^{G}q_{gs}$$

$$q_{gs} \geq 0, \quad g = 1, \ldots, G, \tag{10}$$

$$q_{sk}^{v} \geq 0, \quad k = 1, \ldots, K; v = 1, \ldots, V. \tag{11}$$

We assume that the transaction costs and the operating costs (cf. (9)) are all continuously differentiable and convex, and that the competitive retailers compete in a noncooperative manner. Hence, the optimality conditions for all retailers, simultaneously, under the above assumptions (see also Dafermos and Nagurney (1987), Nagurney, Dong, and Zhang (2002), Dong, Zhang, and Nagurney (2004), and Nagurney et al. (2005)), can be expressed as the following variational inequality: determine $(Q^{2*}, Q^{1*}) \in \mathcal{K}^2$ such that

$$\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{v=1}^{V}\left[\frac{\partial c_{sk}^{v}(Q^{2*})}{\partial q_{sk}^{v}} - \rho_{2sk}^{v*}\right] \times [q_{sk}^{v} - q_{sk}^{v*}]$$

$$+\sum_{g=1}^{G}\sum_{s=1}^{S}\left[\frac{\partial c_s(Q^{1*})}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(Q^{1*})}{\partial q_{gs}} + \rho_{1gs}^{*}\right] \times [q_{gs} - q_{gs}^{*}] \geq 0, \qquad \forall (Q^2, Q^1) \in \mathcal{K}^2, \tag{12}$$

where $\mathcal{K}^2 \equiv \{(Q^2, Q^1) | (Q^2, Q^1) \in R_{+}^{SKV+GS}$ and (8) holds$\}$.

In addition, for notational convenience, we let

$$h_s \equiv \sum_{g=1}^{G} q_{gs}, \quad s = 1, \ldots, S. \tag{13}$$

As defined in Table 1, the operating cost of retailer/supplier $s$, $c_s$, is a function of the total electricity inflows to the retailers, that is:

$$c_s(h) \equiv c_s(Q^1), \quad s = 1, \dots, S. \tag{14}$$

Hence, his marginal cost with respect to $h_s$ is equal to the marginal cost with respect to $q_{gs}$:

$$\frac{\partial c_s(h)}{\partial h_s} \equiv \frac{\partial c_s(Q^1)}{\partial q_{gs}}, \quad s = 1, \dots, S. \tag{15}$$

After the substitution of (13) and (15) into (12) and algebraic simplification, we obtain a variational inequality equivalent to (12), as follows: determine $(h^*, Q^{2*}, Q^{1*}) \in \mathcal{K}^3$ such that

$$\sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{v=1}^{V} \left[ \frac{\partial c_{sk}^v(Q^{2*})}{\partial q_{sk}^v} - \rho_{2sk}^{v*} \right] \times [q_{sk}^v - q_{sk}^{v*}]$$

$$+ \sum_{g=1}^{G} \sum_{s=1}^{S} \left[ \frac{\partial \hat{c}_{gs}(Q^{1*})}{\partial q_{gs}} + \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*] \geq 0, \qquad \forall (Q^1, Q^2, h) \in \mathcal{K}^3, \tag{16}$$

where $\mathcal{K}^3 \equiv \{(h, Q^2, Q^1) | (h, Q^2, Q^1) \in R_+^{S(1+VK+G)} \text{and (8) and (13) hold}\}$.

**Equilibrium Conditions for the Demand Markets**

We now discuss two versions of the equilibrium conditions for the demand markets. In the first case, we assume that the demand functions, $d_k(\rho_3)$, are given, for all demand markets $k = 1, \dots, K$, which will yield the identical demand market equilibrium conditions to those in Nagurney and Matsypura (2004), while in the second case, instead of the demand functions, the inverse demand functions, $\rho_{3k}(d)$; $k = 1, \dots, K$, are assumed known.

**Case 1: Demand Functions are Given**

The assumptions and the demand market equilibrium conditions in this case are identical to those in Nagurney and Matsypura (2004). The consumers take into account the prices charged by the retailers/suppliers and the transaction costs in making their consumption decisions. The equilibrium conditions for consumers at demand market $k$ take the form: for each power retailer $s$; $s = 1, ..., S$ and transaction mode $v$; $v = 1, ..., V$:

$$\rho_{2sk}^{v*} + \hat{c}_{sk}^v(Q^{2*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{sk}^{v*} > 0, \\ \geq \rho_{3k}^*, & \text{if} \quad q_{sk}^{v*} = 0, \end{cases} \tag{17}$$

and

$$d_k(\rho_3^*) \begin{cases} = \sum_{s=1}^S \sum_{v=1}^V q_{sk}^{v*}, & \text{if} \quad \rho_{3k}^* > 0, \\ \leq \sum_{s=1}^S \sum_{v=1}^V q_{sk}^{v*}, & \text{if} \quad \rho_{3k}^* = 0. \end{cases} \tag{18}$$

For notational convenience, we let

$$d_k = \sum_{s=1}^{S} \sum_{v=1}^{V} q_{sk}^v, \quad k = 1, \ldots, K. \tag{19}$$

Conditions (17) state that, in equilibrium, if consumers at demand market $k$ purchase the electricity from retailer $s$ transmitted via mode $v$, then the price the consumers pay is exactly equal to the sum of the price charged by the retailer and the unit transaction cost incurred by the consumers. However, if the price charged by the retailer plus the transaction cost is greater than the price the consumers are willing to pay at the demand market, there will be no transaction between this retailer/demand market pair via that transmission mode.

Conditions (18) state, in turn, that if the equilibrium price the consumers are willing to pay at the demand market is positive, then the total amount of electricity purchased from the power retailers must be equal to the demand at the demand market. If, however, the equilibrium price at the demand market is equal to zero then the total electricity delivered to the demand market may exceed the actual demand.

In equilibrium, conditions (17) and (18) must hold simultaneously for all demand markets. We can also express these equilibrium conditions using the following variational inequality: determine $(Q^{2*}, d^*, \rho_3^*) \in \mathcal{K}^4$, such that

$$\sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{v=1}^{V} \left[\rho_{2sk}^{v*} + \hat{c}_{sk}^v(Q^{2*})\right] \times [q_{sk}^v - q_{sk}^{v*}] - \sum_{k=1}^{K} \rho_{3k}^* \times [d_{3k} - d_{3k}^*] + \sum_{k=1}^{K} [d_k^* - d_k(\rho_3^*)] \times [\rho_{3k} - \rho_{3k}^*] \geq 0,$$

$$\forall (Q^2, d, \rho_3) \in \mathcal{K}^4, \tag{20}$$

where $\mathcal{K}^4 \equiv \{(Q^2, d, \rho_3) | (Q^2, d, \rho_3) \in R_+^{K(SV+2)} \text{and (19) holds}\}$.

**Case 2: Inverse Demand (Demand Market Price) Functions are Given**

Unlike in Case 1, we now assume that the inverse demand (or demand market price) functions are given instead of the demand functions. In Case 2, the market equilibrium conditions at demand market $k$ take the form: for each retailer $s$; $s = 1, ..., S$ and transaction mode $v$; $v = 1, ..., V$:

$$\rho_{2sk}^{v*} + \hat{c}_{sk}^{v}(Q^{2*}) \begin{cases} = \rho_{3k}(d^*), & \text{if} \quad q_{sk}^{v*} > 0, \\ \geq \rho_{3k}(d^*), & \text{if} \quad q_{sk}^{v*} = 0. \end{cases} \tag{21}$$

The interpretation of conditions (21) is the same as that of conditions (17). The equivalent variational inequality takes the form: determine $(Q^{2*}, d^*) \in \mathcal{K}^5$, such that

$$\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{v=1}^{V}\left[\rho_{2sk}^{v*} + \hat{c}_{sk}^{v}(Q^{2*})\right] \times [q_{sk}^{v} - q_{sk}^{v*}] - \sum_{k=1}^{K}\rho_{3k}(d^*) \times [d_k - d_k^*] \geq 0, \quad \forall (Q^2, d) \in \mathcal{K}^5, \tag{22}$$

where $\mathcal{K}^5 \equiv \{(Q^2, d) | (Q^2, d) \in R_+^{K(SV+1)} \text{and (19) holds}\}$.

Deriving formulations for both the case of known demand functions and known demand market price functions allows for greater modeling flexibility and ultimate empirical analysis.

### The Equilibrium Conditions for the Electric Power Supply Chain Network

Next, we construct the equilibrium models for the entire electric power supply chain network. The unknowns of these models include the equilibrium production levels of the generators, $q^*$, the equilibrium transaction flows of the electric power between the generators and the retailers, $Q^{1*}$, the equilibrium transaction flows between the retailers and the consumers at the demand markets, $Q^{2*}$, the equilibrium demands at the consumption markets, $d^*$, and the equilibrium prices at the demand markets, $\rho_3^*$. In equilibrium, the optimality conditions for all the power generators, the optimality conditions for all the retailers/suppliers, and the equilibrium conditions for all the demand markets must be simultaneously satisfied so that no decision-maker can be better off by altering his transactions. We now formally state the equilibrium conditions for the entire electric power supply chain network in the two versions as follows.

### Definition 1a: Electric Power Supply Chain Network Equilibrium – Case 1

*The equilibrium state of the electric power supply chain network of Case 1 is one where the electric power transaction flows between the tiers of the network coincide and the electric power transaction flows and prices satisfy the sum of the optimality conditions for all the power generators (7), the optimality conditions for all the power retailers (16), and the* *equilibrium conditions for all the demand markets (20), so that no decision-maker has any incentive to unilaterally alter his transactions.*

**Definition 1b: Electric Power Supply Chain Network Equilibrium – Case 2**

*The equilibrium state of the electric power supply chain network of Case 2 is one where the electric power transaction flows between the tiers of the network coincide and the electric power transaction flows and prices satisfy the sum of the optimality conditions for all the power generators (7), the optimality conditions for all the power retailers (16), and for the equilibrium conditions for all the demand markets(22), so that no decision-maker has any incentive to unilaterally alter his transactions.*

We now state and prove:

**Theorem 1a: Variational Inequality Formulation of the Electric Power Supply Chain Network Equilibrium – Case 1**

*The equilibrium conditions governing the electric power supply chain network according to Definition 1a coincide with the solution of the variational inequality given by: determine* $(q^*, h^*, Q^{1*}, Q^{2*}, d^*, \rho_3^*) \in \mathcal{K}^6$ *satisfying:*

$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{g=1}^{G}\sum_{s=1}^{S} \left[ \frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(Q^{1*})}{\partial q_{gs}} \right] \times [q_{gs} - q_{gs}^*]$$

$$+ \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{v=1}^{V} \left[ \frac{\partial c_{sk}^v(Q^{2*})}{\partial q_{sk}^v} + \hat{c}_{sk}^v(Q^{2*}) \right] \times [q_{sk}^v - q_{sk}^{v*}] - \sum_{k=1}^{K} \rho_{3k}^* \times [d_k - d_k^*]$$

$$+ \sum_{k=1}^{K} [d_k^* - d_k(\rho_3^*)] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall (q, h, Q^1, Q^2, d, \rho_3) \in \mathcal{K}^6, \tag{23}$$

$$\textit{where } \mathcal{K}^6 \equiv \{(q, h, Q^1, Q^2, d, \rho_3) | (q, h, Q^1, Q^2, d, \rho_3) \in R_+^{G+S+GS+VSK+2K}$$

$$\textit{and } (1),\ (8),\ (13),\ \textit{and } (19)\ \textit{hold}\}.$$

**Proof:** We first prove that an equilibrium according to Definition 1a coincides with the solution of variational inequality (23). Indeed, summation of (7), (16), and (20), after algebraic simplifications, yields (23).

We now prove the converse, that is, a solution to variational inequality (23) satisfies the sum of conditions (7), (16), and (20), and is, therefore, an electric power network equilibrium pattern according to Definition 1a.

First, we add the term $\rho_{1gs}^* - \rho_{1gs}^*$ to the first term in the third summand expression in (23). Then, we add the term $\rho_{2sk}^{v*} - \rho_{2sk}^{v*}$ to the first term in the fourth summand expression in (23). Since these terms are all equal to zero, they do not change (23). Hence, we obtain the following inequality:

$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*]$$
$$+ \sum_{g=1}^{G}\sum_{s=1}^{S} \left[ \frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(Q^{1*})}{\partial q_{gs}} + \rho_{1gs}^* - \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*]$$
$$+ \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{v=1}^{V} \left[ \frac{\partial c_{sk}^v(Q^{2*})}{\partial q_{sk}^v} + \hat{c}_{sk}^v(Q^{2*}) + \rho_{2sk}^{v*} - \rho_{2sk}^{v*} \right] \times [q_{sk}^v - q_{sk}^{v*}]$$
$$- \sum_{k=1}^{K} \rho_{3k}^* \times [d_k - d_k^*] + \sum_{k=1}^{K} [d_k^* - d_k(\rho_3^*)] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall (q,h,Q^1,Q^2,d,\rho_3) \in \mathcal{K}^6, \quad (24)$$

which can be rewritten as:

$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{g=1}^{G}\sum_{s=1}^{S} \left[ \frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}} - \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*]$$
$$+ \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{v=1}^{V} \left[ \frac{\partial c_{sk}^v(Q^{2*})}{\partial q_{sk}^v} - \rho_{2sk}^{v*} \right] \times [q_{sk}^v - q_{sk}^{v*}]$$
$$+ \sum_{s=1}^{S}\sum_{g=1}^{G} \left[ \frac{\partial \hat{c}_{gs}(Q^{1*})}{\partial q_{gs}} + \rho_{1gs}^* \right] \times [q_{gs} - q_{gs}^*]$$
$$+ \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{v=1}^{V} \left[ \rho_{2sk}^{v*} + \hat{c}_{sk}^v(Q^{2*}) \right] \times [q_{sk}^v - q_{sk}^{v*}]$$
$$- \sum_{k=1}^{K} \rho_{3k}^* \times [d_k - d_k^*] + \sum_{k=1}^{K} [d_k^* - d_k(\rho_3^*)] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall (q,h,Q^1,Q^2,d,\rho_3) \in \mathcal{K}^6. \quad (25)$$

Clearly, (25) is the sum of the optimality conditions (7) and (16), and the equilibrium conditions (20) and is, hence, according to Definition 1a an electric power network quilibrium.
□

We now describe how to recover the prices associated with the first two tiers of nodes in the electric power supply chain network. Clearly, the components of the vector $\rho_3^*$ can be directly obtained from the solution to variational inequality (23). We now describe how to recover the prices $\rho_{1gs}^*$, for all $g, s$, and $\rho_{2sk}^{v*}$ for all $s, k, v$, from the solution of variational inequality (23). The prices $\rho_2^*$ associated with the power retailers, in turn, can be obtained by setting (cf. (17)) $\rho_{2sk}^{v*} = \rho_{3k}^* - \hat{c}_{sk}^v(Q^{2*})$ for any $s, v, k$ such that $q_{sk}^{v*} > 0$. The prices $\rho_1^*$, in turn, can be recovered by setting (cf. (4)) $\rho_{1gs}^* = \frac{\partial f_g(Q^{1*})}{\partial q_{gs}} + \frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}}$ for any $g, s$ such that $q_{gs}^* > 0$.

Now we state the following:

**Theorem 1b: Variational Inequality Formulation of the Electric Power Supply Chain Network Equilibrium – Case 2**

*The equilibrium conditions governing the electric power supply chain network according to Definition 1b coincide with the solution of the variational inequality given by: determine* $(q^*, h^*, Q^{1*}, Q^{2*}, d^*) \in \mathcal{K}^7$ *satisfying:*

$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q_g^*] + \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h_s^*] + \sum_{g=1}^{G} \sum_{s=1}^{S} \left[ \frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(Q^{1*})}{\partial q_{gs}} \right] \times [q_{gs} - q_{gs}^*]$$

$$+ \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{v=1}^{V} \left[ \frac{\partial c_{sk}^v(Q^{2*})}{\partial q_{sk}^v} + \hat{c}_{sk}^v(Q^{2*}) \right] \times [q_{sk}^v - q_{sk}^{v*}] - \sum_{k=1}^{K} \rho_{3k}(d^*) \times [d_k - d_k^*] \geq 0,$$

$$\forall (q, h, Q^1, Q^2, d) \in \mathcal{K}^7, \tag{26}$$

*where*

$$\mathcal{K}^7 \equiv \{(q, h, Q^1, Q^2, d) | (q, h, Q^1, Q^2, d) \in R_+^{G+S+GS+VSK+K}$$
$$\textit{and } (1), (8), (13), \textit{ and } (19) \textit{ hold.}\}$$

**Proof:** Theorem 1b can be easily proved using similar arguments to those in the proof of Theorem 1a.

In Case 2, the prices associated with the first two tiers of nodes in the electric power supply chain network can be recovered in a similar way to that in Case 1. □

Nagurney and Matsypura (2004) derived a variational inequality formulation of electric power supply chain network equilibrium in the case of known demand functions but since the conservation of flow expression (8) in their model was an inequality the formulation also had Lagrange multipliers reflecting nodal prices associated with those inequalities as variables in their variational inequality. Of course, in the case in which the nodal prices/Lagrange multipliers are positive in equilibrium then the electric power flow out of each competitive retailer would be equal to the electric power flow in and the variational inequality obtained in Nagurney and Matsypura (2004) could be cast into the form (23). In Section 5, we provide numerical examples in which this is, indeed, the case when we solve several examples from Nagurney and Matsypura (2004) but formulated as elastic demand transportation network equilibrium problems.

## 3. The Transportation Network Equilibrium Models with Elastic Demands

In this Section, we review two versions of transportation network equilibrium models with elastic demands, one with given demand functions (see Dafermos and Nagurney (1984a)) and the other with given travel disutility functions (see Dafermos (1982)).

### Transportation Network Equilibrium Model with Demand Functions

We consider a network $\mathcal{G}$ with the set of links $L$ with $K$ elements, the set of paths $P$ with $Q$ elements, and the set of origin/destination (O/D) pairs $W$ with $Z$ elements. We denote the set of paths joining O/D pair $w$ by $P_w$. Links are denoted by $a, b$, etc; paths by $p, q$, etc., and O/D pairs by $w_1$, $w_2$, etc.

We denote the flow on path $p$ by $x_p$ and the flow on link $a$ by $f_a$. The user travel cost on a path $p$ is denoted by $C_p$ and the user travel cost on a link $a$ by $c_a$. We denote the travel demand associated with traveling between O/D pair $w$ by $d_w$ and the travel disutility by $\lambda_w$.

The link flows are related to the path flows through the following conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{27}$$

where $\delta_{ap} = 1$ if link $a$ is contained in path $p$, and $\delta_{ap} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on paths that contain that link.

The user costs on paths are related to user costs on links through the following equations:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P, \tag{28}$$

that is, the user cost on a path is equal to the sum of user costs on links that make up the path.

For the sake of generality, we allow the user cost on a link to depend upon the entire vector of link flows, denoted by $f$, so that

$$c_a = c_a(f), \quad \forall a \in L. \tag{29}$$

We assume travel demand functions, such that

$$d_w = d_w(\lambda), \quad \forall w \in W, \tag{30}$$

where $\lambda$ is the vector of travel disutilities with the travel disutility associated with O/D pair being denoted by $\lambda_w$.

**Definition 2a: Transportation Network Equilibrium – Known Demand Functions**

*As shown in Dafermos and Nagurney (1984a); see also Aashtiani and Magnanti (1981), Fisk and Boyce (1983), Nagurney and Zhang (1996), and Nagurney (1993), a path flow and disutility pattern* $(x^*, \lambda^*) \in R_+^{Q+Z}$ *is said to be a transportation network equilibrium, if, once established, no user has any incentive to alter his travel decisions. The state can be expressed by the following equilibrium conditions which must hold for every O/D pair* $w \in W$ *and every path* $p \in P_w$*:*

$$C_p(x^*) - \lambda_w^* \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0, \end{cases} \tag{31}$$

*and*

$$\sum_{p \in P_w} x_p^* \begin{cases} = d_w(\lambda^*), & \text{if} \quad \lambda_w^* > 0, \\ \geq d_w(\lambda^*), & \text{if} \quad \lambda_w^* = 0. \end{cases} \tag{32}$$

Conditions (31) state that all utilized paths connecting an O/D pair have equal and minimal user costs and these costs are equal to the travel disutility associated with traveling between that O/D pair. Conditions (32) state that the market clears for each O/D pair under a positive price or travel disutility. As described in Dafermos and Nagurney (1984a) these transportation network equilibrium conditions (31) and (32) can be formulated as the following variational inequality in path flows: determine $(x^*, \lambda^*) \in R_+^{Q+Z}$ such that

$$\sum_{w \in W} \sum_{p \in P_w} [C_p(x^*) - \lambda_w^*] \times \left[x_p - x_p^*\right] + \sum_{w \in W} \sum_{p \in P_w} \left[x_p^* - d_w(\lambda^*)\right] \times [\lambda_w - \lambda_w^*] \geq 0, \quad \forall (x, \lambda) \in R_+^{Q+Z}. \tag{33}$$

Now we also provide the equivalent variational inequality in link flows due to Dafermos and Nagurney (1984a). For additional background, see the book by Nagurney (1993).

**Theorem 2a**

*A link flow pattern and associated travel demand and disutility pattern is a transportation network equilibrium if and only if it satisfies the variational inequality problem: determine* $(f^*, d^*, \lambda^*) \in \mathcal{K}^8$ *satisfying*

$$\sum_{a \in L} c_a(f^*) \times (f_a - f_a^*) - \sum_{w \in W} \lambda_w^* \times (d_w - d_w^*) + \sum_{w \in W} [d_w^* - d_w(\lambda^*)] \times [\lambda_w - \lambda_w^*] \geq 0, \quad \forall (f, d, \lambda) \in \mathcal{K}^8, \tag{34}$$

*where* $\mathcal{K}^8 \equiv \{(f, d, \lambda) \in R_+^{K+2Z} |$ *there exists an* $x$ *satisfying* (27) *and* $d_w = \sum_{p \in P_w} x_p, \forall w\}$.

**Transportation Network Equilibrium Model with Disutility Functions**

We now recall the transportation network equilibrium model with disutility functions due to Dafermos (1982). The notation is the same as in the model with known demand functions. However, we now assume that the following conservation of flow equations:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w. \tag{35}$$

Also, we assume now, as given, disutility functions, such that

$$\lambda_w = \lambda_w(d), \quad \forall w, \tag{36}$$

where $d$ is the vector of travel demands with travel demand associated with O/D pair $w$ being denoted by $d_w$.

**Definition 2b: Transportation Network Equilibrium – Known Travel Disutility Functions**

*In equilibrium, the following conditions must hold for each O/D pair* $w \in W$ *and each path* $p \in P_w$*:*

$$C_p(x^*) - \lambda_w(d^*) \begin{cases} = 0, & \text{if} \quad x_p^* > 0, \\ \geq 0, & \text{if} \quad x_p^* = 0. \end{cases} \tag{37}$$

The interpretation of conditions (37) is the same as that of the conditions (31). As proved in Dafermos (1982), the transportation network equilibrium conditions (37) are equivalent to the following variational inequality: determine $(x^*, d^*) \in \mathcal{K}^9$ such that

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^*\right] - \sum_{w \in W} \lambda_w(d^*) \times [d_w - d_w^*] \geq 0, \quad \forall (x, d) \in \mathcal{K}^9, \tag{38}$$

where $\mathcal{K}^9 \equiv \{(x, d) | (x, d) \in R_+^{K+Z} \text{and } d_w = \sum_{p \in P_w} x_p, \forall w\}$.

We now recall the equivalent variational inequality in link form due to Dafermos (1982).

**Theorem 2b**

*A link flow pattern and associated travel demand pattern is a transportation network equilibrium if and only if it satisfies the variational inequality problem: determine* $(f^*, d^*) \in \mathcal{K}^{10}$ *satisfying*

$$\sum_{a \in L} c_a(f^*) \times (f_a - f_a^*) - \sum_{w \in W} \lambda_w(d^*) \times (d_w - d_w^*) \geq 0, \quad \forall (f, d) \in \mathcal{K}^{10}, \tag{39}$$

*where* $\mathcal{K}^{10} \equiv \{(f, d) \in R_+^{K+Z} |$ *there exists an* $x$ *satisfying* (27) *and* $d_w = \sum_{p \in P_w} x_p, \forall w\}$.

## 4. Transportation Network Equilibrium Reformulations of the Electric Power Supply Chain Network Equilibrium Models

In this Section, we show that each of the two electric power supply chain network equilibrium models presented in Section 2 is isomorphic to a properly configured transportation network equilibrium model through the establishment of a supernetwork equivalence of the former. In particular, we will first establish the supernetwork equivalence of the electric power supply chain network equilibrium model with known demand functions and a properly constructed elastic demand transportation network equilibrium model. We will then show that the electric power supply chain network equilibrium model of the second case is also equivalent to a properly configured elastic demand transportation network equilibrium model but with known disutility functions.

### Supernetwork Equivalence of the Electric Power Supply Chain Network in Case 1

We now establish the supernetwork equivalence of the electric power supply chain network equilibrium model in Case 1 to the transportation network equilibrium model with known demand functions over a particular network.

Consider an electric power supply chain network as discussed in Section 2 with given power generators: $g = 1, \ldots, G$; retailers/suppliers: $s = 1, \ldots, S$; transmission service providers: $v = 1, \ldots, V$, and demand markets: $k = 1, \ldots, K$. The supernetwork, $\mathcal{G}_{\mathcal{S}}$, of the isomorphic transportation network equilibrium model is depicted in Figure 2 and is constructed as follows. It consists of five tiers of nodes with the origin node 0 at the top or first tier and the destination nodes at the fifth or bottom tier. Specifically, $\mathcal{G}_{\mathcal{S}}$ consists of a single origin node 0 at the first tier, and $K$ destination nodes at the bottom tier, denoted, respectively, by: $z_1, \ldots, z_K$. There are $K$ O/D pairs in $\mathcal{G}_{\mathcal{S}}$ denoted by $w_1 = (0, z_1)$, …, $w_k = (0, z_k)$,…, $w_K = (0, z_K)$. Node 0 is connected to each second-tiered node $x_g$; $g = 1, \ldots, G$ by a single link. Each second-tiered node $x_g$, in turn, is connected to each third-tiered node $y_s$; $s = 1, \ldots, S$ by a single link. Each third tiered-node $y_s$ is connected to the corresponding fourth-tiered node $y_{s'}$ by a single link. Finally, each fourth-tiered node $y_{s'}$ is connected to each destination node $z_k$; $k = 1, \ldots, K$ at the fifth tier by $V$ parallel links.

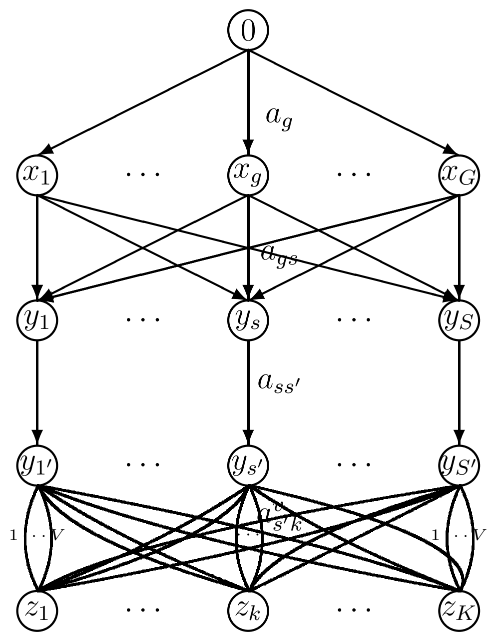


Figure 2: The $\mathcal{G}_{\mathcal{S}}$ Supernetwork Representation of Electric Power Supply Chain Network Equilibrium

Hence, in $\mathcal{G}_{\mathcal{S}}$, there are $G+2S+K+1$ nodes, $G+GS+S+SKV$ links, $K$ O/D pairs, and $GSKV$ paths. We now define the link and link flow notation. Let $a_g$ denote the link from node 0 to node $x_g$ with associated link flow $f_{a_g}$, for $g=1,\ldots,G$. Let $a_{gs}$ denote the link from node $x_g$ to node $y_s$ with associated link flow $f_{a_{gs}}$ for $g=1,\ldots,G$ and $s=1,\ldots,S$. Also, let $a_{ss'}$ denote the link connecting node $y_s$ with node $y_{s'}$ with associated link flow $f_{a_{ss'}}$ for $s$; $s'=1,\ldots,S$. Finally, let $a^v_{s'k}$ denote the $v$-th link joining node $y_{s'}$ with node $z_k$ for $s'=1',\ldots,S'$; $v=1,\ldots,V$, and $k=1,\ldots,K$ and with associated link flow $f_{a^v_{s'k}}$. We group the link flows into the vectors as follows: we group the $\{f_{a_g}\}$ into the vector $f^1$; the $\{f_{a_{sg}}\}$ into the vector $f^2$; the $\{f_{a_{ss'}}\}$ into the vector $f^3$, and the $\{f_{a^v_{s'k}}\}$ into the vector $f^4$.

Thus, a typical path connecting O/D pair $w_k=(0,z_k)$, is denoted by $p^v_{gss'k}$ and consists of four links: $a_g, a_{gs}, a_{ss'}$, and $a^v_{s'k}$. The associated flow on the path is denoted by $x_{p^v_{gss'k}}$. Finally, we let $d_{w_k}$ be the demand associated with O/D pair $w_k$ where $\lambda_{w_k}$ denotes the travel disutility for $w_k$.

Note that the following conservation of flow equations must hold on the network $\mathcal{G}_{\mathcal{S}}$:

$$f_{a_g}=\sum_{s=1}^{S}\sum_{s'=1}^{S'}\sum_{k=1}^{K}\sum_{v=1}^{V}x_{p^v_{gss'k}}, \quad g=1,\ldots,G, \tag{40}$$

$$f_{a_{gs}}=\sum_{s'=1}^{S'}\sum_{k=1}^{K}\sum_{v=1}^{V}x_{p^v_{gss'k}}, \quad g=1,\ldots,G; s=1,\ldots,S, \tag{41}$$

$$f_{a_{ss'}}=\sum_{g=1}^{G}\sum_{k=1}^{K}\sum_{v=1}^{V}x_{p^v_{gss'k}}, \quad ss'=11',\ldots,SS', \tag{42}$$

$$f_{a^v_{s'k}}=\sum_{g=1}^{G}\sum_{s=1}^{S}x_{p^v_{gss'k}}, \quad s'=1,\ldots,S'; v=1,\ldots,V; k=1,\ldots,K. \tag{43}$$

Also, we have that

$$d_{w_k}=\sum_{g=1}^{G}\sum_{s=1}^{S}\sum_{s'=1}^{S'}\sum_{v=1}^{V}x_{p^v_{gss'k}}, \quad k=1,\ldots,K. \tag{44}$$

If all path flows are nonnegative and (40)–(44) are satisfied, the feasible path flow pattern induces a feasible link flow pattern.

We can construct a feasible link flow pattern for $\mathcal{G}_\mathcal{S}$ based on the corresponding feasible electric power flow pattern in the electric power supply chain network model, $(q, h, Q^1, Q^2, d, \rho_3) \in \mathcal{K}^6$, in the following way:

$$q_g \equiv f_{a_g}, \quad g = 1, \ldots, G, \tag{45}$$

$$q_{gs} \equiv f_{a_{gs}}, \quad g = 1, \ldots, G; s = 1, \ldots, S, \tag{46}$$

$$h_s \equiv f_{a_{ss'}}, \quad s = 1, \ldots, S; s' = 1', \ldots, S', \tag{47}$$

$$q^v_{sk} = f_{a^v_{s'k}}, \quad s' = 1', \ldots, S'; v = 1, \ldots, V; k = 1, \ldots, K, \tag{48}$$

$$d_k = \sum_{s=1}^{S} \sum_{v=1}^{V} q^v_{sk}, \quad k = 1, \ldots, K. \tag{49}$$

Note that if $(q, Q^1, h, Q^2, d, \rho_3)$ is feasible then the link flow pattern constructed according to (45) – (49) is also feasible and the corresponding path flow pattern which induces this link flow (and demand) pattern is also feasible.

We now assign user (travel) costs on the links of the network $\mathcal{G}_\mathcal{S}$ as follows: with each link $a_g$ we assign a user cost $c_{a_g}$ defined by

$$c_{a_g} \equiv \frac{\partial f_g}{\partial q_g}, \quad g = 1, \ldots, G, \tag{50}$$

with each link $a_{gs}$ we assign a user cost $c_{a_{gs}}$ defined by:

$$c_{a_{gs}} \equiv \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}}, \quad g = 1, \ldots, G; s = 1, \ldots, S, \tag{51}$$

with each link $ss'$ we assign a user cost defined by

$$c_{a_{ss'}} \equiv \frac{\partial c_s}{\partial h_s}, \quad ss' = 11', \ldots, SS'. \tag{52}$$

Finally, for each link $a^v_{s'k}$ we assign a user cost defined by

$$c_{a^v_{s'k}} \equiv \frac{\partial c^v_{sk}}{\partial q^v_{sk}} + \hat{c}^v_{sk}, \quad s = 1, \ldots, S; s' = 1, \ldots, S'; v = 1, \ldots, V; k = 1, \ldots, K. \tag{53}$$

Then a user of path $p^v_{gss'k}$, for $g = 1,\ldots,G$; $s = 1,\ldots,S$; $s' = 1',\ldots,S'$; $v = 1,\ldots,V$; $k = 1,\ldots,K$, on network $\mathcal{G}_\mathcal{S}$ in Figure 2 experiences a path cost $C_{p^v_{gss'k}}$ given by

$$C_{p^v_{gss'k}} = c_{a_g} + c_{a_{sg}} + c_{a_{ss'}} + c_{a^v_{s'k}} = \frac{\partial f_g}{\partial q_g} + \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}} + \frac{\partial c_s}{\partial h_s} + \frac{\partial c^v_{sk}}{\partial q^v_{sk}} + \hat{c}^v_{sk}. \tag{54}$$

Also, we assign the (travel) demands associated with the O/D pairs as follows:

$$d_{w_k} \equiv d_k, \quad k = 1,\ldots,K, \tag{55}$$

and the (travel) disutilities:

$$\lambda_{w_k} \equiv \rho_{3k}, \quad k = 1,\ldots,K. \tag{56}$$

Consequently, the equilibrium conditions (31) and (32) for the transportation network equilibrium model on the network $\mathcal{G}_\mathcal{S}$ state that for every O/D pair $w_k$ and every path connecting the O/D pair $w_k$:

$$C_{p^v_{gss'k}} - \lambda^*_{w_k} = \frac{\partial f_g}{\partial q_g} + \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}} + \frac{\partial c_s}{\partial h_s} + \frac{\partial c^v_{sk}}{\partial q^v_{sk}} + \hat{c}^v_{sk} - \lambda^*_{w_k} \begin{cases} = 0, & \text{if} \quad x^*_{p^v_{gss'k}} > 0, \\ \geq 0, & \text{if} \quad x^*_{p^v_{gss'k}} = 0, \end{cases} \tag{57}$$

and

$$\sum_{p \in P_{w_k}} x^*_{p^v_{gss'k}} \begin{cases} = d_{w_k}(\lambda^*), & \text{if} \quad \lambda^*_{w_k} > 0, \\ \geq d_{w_k}(\lambda^*), & \text{if} \quad \lambda^*_{w_k} = 0. \end{cases} \tag{58}$$

We now show that the variational inequality formulation of the equilibrium conditions (57) and (58) in link form as in (34) is equivalent to the variational inequality (23) governing the electric power network equilibrium. For the transportation network equilibrium problem on $\mathcal{G}_\mathcal{S}$, according to Theorem 2a, we have that a link flow, travel demand, and travel disutility pattern $(f^*, d^*, \lambda^*) \in \mathcal{K}^8$ is an equilibrium (according to (57) and (58)), if and only if it satisfies the variational inequality:

$$\sum_{g=1}^{G} c_{a_g}(f^{1*}) \times (f_{a_g} - f^*_{a_g}) + \sum_{g=1}^{G}\sum_{s=1}^{S} c_{a_{gs}}(f^{2*}) \times (f_{a_{gs}} - f^*_{a_{gs}})$$

$$+ \sum_{ss'=11'}^{SS'} c_{a_{ss'}}(f^{3*}) \times (f_{a_{ss'}} - f^*_{a_{ss'}}) + \sum_{s'=1}^{S'}\sum_{k=1}^{K}\sum_{v=1}^{V} c_{a^v_{s'k}}(f^{4*}) \times (f_{a^v_{s'k}} - f^*_{a^v_{s'k}})$$

$$-\sum_{k=1}^{K}\lambda^*_{w_k}\times(d_{w_k}-d^*_{w_k})+\sum_{k=1}^{K}\left[d^*_{w_k}-d_{w_k}(\lambda^*)\right]\times\left[\lambda_{w_k}-\lambda^*_{w_k}\right]\geq 0,\quad\forall(f,d,\lambda)\in\mathcal{K}^8. \quad (59)$$

After the substitution of (45)–(49) and (50)–(53), we have the following variational inequality: determine $(q^*,h^*,Q^{1*},Q^{2*},d^*,\rho^*_3)\in\mathcal{K}^6$ satisfying:

$$\sum_{g=1}^{G}\frac{\partial f_g(q^*)}{\partial q_g}\times[q_g-q^*_g]+\sum_{s=1}^{S}\frac{\partial c_s(h^*)}{\partial h_s}\times[h_s-h^*_s]+\sum_{g=1}^{G}\sum_{s=1}^{S}\left[\frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}}+\frac{\partial\hat{c}_{gs}(Q^{1*})}{\partial q_{gs}}\right]\times[q_{gs}-q^*_{gs}]$$

$$+\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{v=1}^{V}\left[\frac{\partial c^v_{sk}(Q^{2*})}{\partial q^v_{sk}}+\hat{c}^v_{sk}(Q^{2*})\right]\times[q^v_{sk}-q^{v*}_{sk}]-\sum_{k=1}^{K}\rho^*_{3k}\times[d_k-d^*_k]$$

$$+\sum_{k=1}^{K}[d^*_k-d_k(\rho^*_3)]\times[\rho_{3k}-\rho^*_{3k}]\geq 0,\quad\forall(q,h,Q^1,Q^2,d,\rho_3)\in\mathcal{K}^6. \quad (60)$$

Variational inequality (60) is precisely variational inequality (23) governing the electric power supply chain network equilibrium. Hence, we have the following result:

**Theorem 3a**

*A solution $(q^*,h^*,Q^{1^*},Q^{2^*},d^*,\rho^*_3)\in\mathcal{K}^6$ of the variational inequality (23) governing the electric power supply chain network equilibrium coincides with the (via (45) – (49) and (55) – (56)) feasible link flow, travel demand, and travel disutility pattern for the supernetwork $\mathcal{G}_{\mathcal{S}}$ constructed above and satisfies variational inequality (34). Hence, it is a traffic network equilibrium according to Theorem 2a.*

**Supernetwork Equivalence of the Electric Power Supply Chain Network in Case 2**

Using the method demonstrated above, we can easily recover the supernetwork equivalence of the electric power supply chain network model in the second case. Indeed, we may use the same $\mathcal{G}_{\mathcal{S}}$ network configuration except that we define $\lambda_{w_k}(d)$, the disutility function of O/D pair $k$; $k=1,\ldots,K$ as follows:

$$\lambda_{w_k}(d)\equiv\rho_{3k}(d). \quad (61)$$

Consequently, according to the equilibrium conditions (37) for the transportation network equilibrium model with known disutility functions, we state that in equilibrium, for every

O/D pair $w_k$ and every path connecting the O/D pair $w_k$:

$$C_{p^v_{gss'k}} - \lambda_{w_k}(d^*) = \frac{\partial f_g}{\partial q_g} + \frac{\partial c_{gs}}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}}{\partial q_{gs}} + \frac{\partial c_s}{\partial h_s} + \frac{\partial c^v_{sk}}{\partial q^v_{sk}} + \hat{c}^v_{sk} - \rho_{3k}(d^*) \begin{cases} = 0, & \text{if} \quad x^*_{p^v_{gss'k}} > 0, \\ \geq 0, & \text{if} \quad x^*_{p^v_{gss'k}} = 0. \end{cases} \tag{62}$$

After the usage of the link form variational inequality (39) and the substitution of (45)–(53), we have that:

$$\sum_{g=1}^{G} \frac{\partial f_g(q^*)}{\partial q_g} \times [q_g - q^*_g] + \sum_{s=1}^{S} \frac{\partial c_s(h^*)}{\partial h_s} \times [h_s - h^*_s] + \sum_{g=1}^{G} \sum_{s=1}^{S} \left[ \frac{\partial c_{gs}(Q^{1*})}{\partial q_{gs}} + \frac{\partial \hat{c}_{gs}(Q^{1*})}{\partial q_{gs}} \right] \times [q_{gs} - q^*_{gs}]$$

$$+ \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{v=1}^{V} \left[ \frac{\partial c^v_{sk}(Q^{2*})}{\partial q^v_{sk}} + \hat{c}^v_{sk}(Q^{2*}) \right] \times [q^v_{sk} - q^{v*}_{sk}] - \sum_{k=1}^{K} \rho_{3k}(d^*) \times [d_k - d^*_k] \geq 0,$$

$$\forall (q, h, Q^1, Q^2, d) \in \mathcal{K}^7. \tag{63}$$

Variational inequality (63) is precisely variational inequality (26) governing the electric power network equilibrium in Case 2. Hence, we reach the following conclusion:

**Theorem 3b**

*A solution $(q^*, h^*, Q^{1^*}, Q^{2^*}, d^*) \in \mathcal{K}^7$ of the variational inequality (26) governing an electric power supply chain network equilibrium in Case 2 coincides with the (via (45) – (49) and (55) – (56)) feasible link flow and travel demand for the supernetwork $\mathcal{G}_\mathcal{S}$ constructed above and satisfies variational inequality (39). Hence, it is a transportation network equilibrium according to Theorem 2b.*

We now further discuss the interpretation of the electric power supply chain network equilibrium conditions (57) in Case 1 and (62) in Case 2. These conditions define the electric power supply chain network equilibrium in terms of paths and path flows, which, as shown above, coincide with Wardrop's (1952) first principle of user-optimization in the context of transportation networks over the network given in Figure 2. Hence, we now have an entirely new interpretation of electric power supply chain network equilibrium which states that only minimal cost paths will be used from the super source node 0 to any destination node. Moreover, the cost on the utilized paths for a particular O/D pair is equal to the disutility (or the demand market price) that the users are willing to pay. This interpretation also implies a type of efficiency principle regarding electric power supply chain network operation and utilization.

In Section 5, we will show how Theorems 3a and 3b can be utilized to exploit algorithmically the theoretical results obtained above when we compute the equilibrium patterns of numerical electric power supply chain network examples using an algorithm previously used for the computation of elastic demand transportation network equilibria. Of course, existence and uniqueness results obtained for elastic demand transportation network equilibrium models as in Dafermos (1982) and Dafermos and Nagurney (1984a) as well as stability and sensitivity analysis results (see also Nagurney and Zhang (1996)) can now be transferred to electric power networks using the formalism/equivalence established above.

It is also important to emphasize that the connection formalized above between electric power supply chain networks and transportation networks also unveils opportunities for further modeling enhancements. For example, one may construct network representations of actual power grids and substitute these for the corresponding transmission links in the supernetwork. The concept of equilibrium path flows would still be appropriate and relevant but with the supernetwork expanded accordingly. For example, an analogous extension but in the case of spatial price network equilibrium problems can be found in Dafermos and Nagurney (1984b).

## 5. Computations

In this Section, we provide numerical examples to demonstrate how the theoretical results in this paper can be applied in practice. In the first three examples, we consider numerical electric power supply chain network problems with given demand functions that had been solved earlier by Nagurney and Matsypura (2004). We then provide an additional numerical example, also taken from Nagurney and Matsypura (2004), in which the results are given in the case of known demand functions and since the inverses are readily obtained also for that case.

We utilize the Euler method for our numerical computations. The Euler method is induced by the general iterative scheme of Dupuis and Nagurney (1993) and has been applied by Zhang and Nagurney (1997) to solve the variational inequality problem (33) in path flows (or, equivalently, variational inequality (34) in link flows). In addition, it has been applied by Nagurney and Zhang (1996) to solve variational inequality (38) in path flows (equivalently, variational inequality (39) in link flows).

### Euler Method – Case 1

For the solution of (33), the Euler method takes the form: at iteration $\tau$ compute the path flows for paths $p \in P$ according to:

$$x_p^{\tau+1} = \max\{0, x_p^\tau + \alpha_\tau(\lambda_w^\tau - C_p(x^\tau))\}, \tag{64}$$

and the travel disutilities for all O/D pairs $w \in W$ according to:

$$\lambda_w^{\tau+1} = \max\{0, \lambda_w^\tau + \alpha_\tau(d_w(\lambda^\tau) - \sum_{p \in P_w} x_p^\tau)\}, \tag{65}$$

where $\{a_\tau\}$ is a sequence of positive real numbers that satisfies: $\lim_{\tau \to 0}$ and $\sum_{\tau=1}^{\infty} \alpha_\tau = \infty$. Such a sequence is required for convergence (see Nagurney and Zhang (1996)).

The elegance of this computational procesure lies in that, at each iteration, according to (64) and (65), the path flows and the travel disutilities are computed explicitly in closed form.

**Euler Method – Case 2**

For the solution of (38), the Euler method takes the form: at iteration $\tau$ compute the path flows for paths $p \in P$ (and the travel demands) according to:

$$x_p^{\tau+1} = \max\{0, x_p^\tau + \alpha_\tau(\lambda_w(d^\tau) - C_p(x^\tau))\}. \tag{66}$$

The simplicity of (66) lies in the explicit formula that allows for the computation of the path flows in closed form at each iteration. The demands at each iteration simply satisfy (35) and this expression can be substituted into the $\lambda_w(\cdot)$ functions.

The Euler method (both the Case 1 and the Case 2 versions) was implemented in FORTRAN and the computer system used was a Sun system at the University of Massachusetts at Amherst. For the first three examples the convergence criterion utilized was that the absolute value of the path flows and travel disutilities between two successive iterations differed by no more than $10^{-4}$. For the fourth and final example in the case of known travel disutility (or demand market price functions) the convergence criterion utilized was that the absolute value of the path flows between two successive iterations differed by no more than $10^{-4}$. The sequence $\{\alpha_\tau\}$ in the Euler method (cf. (64) and (65), as well as (66)) was set to: $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$. The Euler method in both versions was initialized by setting the demands equal to 100 for each O/D pair with the path flows equally distributed and the travel disutilities all set equal to 1.

In all the numerical examples, the electric power supply chain network consisted of three power generators, two retailers/suppliers, one transmission provider, and three demand markets as depicted in Figure 3. The supernetwork representation which allows for the transformation (as proved in Section 4) to a transportation network equilibrium problem is given also in Figure 3. Hence, in the numerical examples (see also Figure 2) we had that: $G = 3$, S=2, $S' = 2'$, and $K = 3$.

The notation is presented here and in the subsequent examples in the form of the appropriate electric power supply chain network equilibrium model of Section 2. We then provide the complete supernetwork representation in terms of O/D pairs, paths, etc. The translations of the equilibrium path flows, link flows, and travel disutilities into the equilibrium

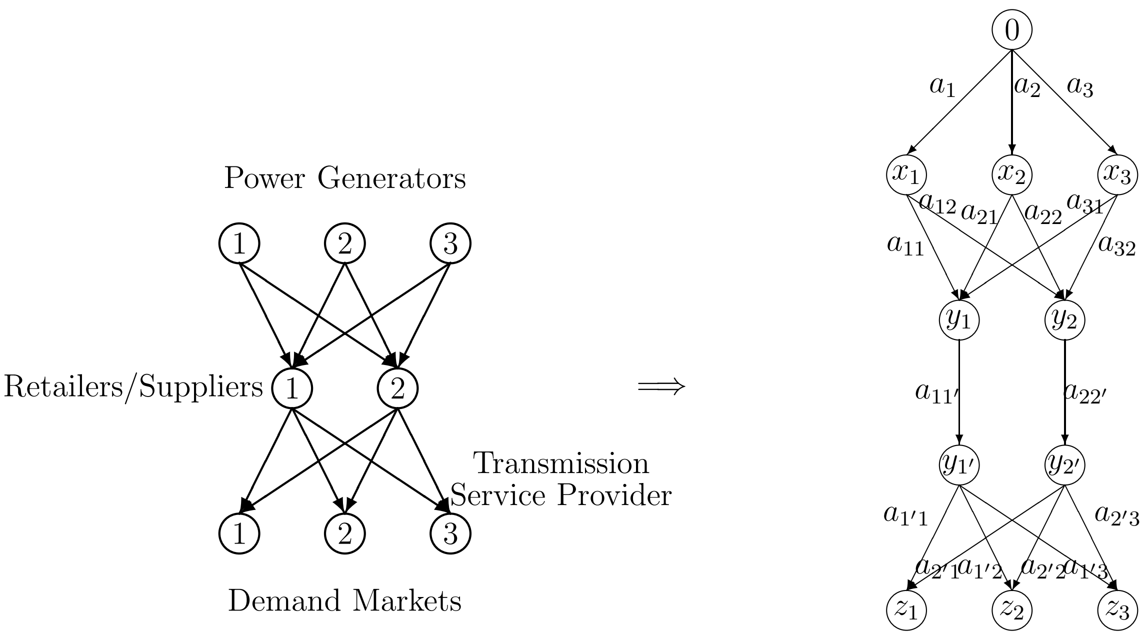


Figure 3: Electric Power Supply Chain Network and Corresponding Supernetwork $\mathcal{G}_{\mathcal{S}}$ for the Numerical Examples

electric power flows and prices is then given, for completeness and easy reference.

**Example 1**

For completeness, the data for this numerical example, due to Nagurney and Matsypura (2004), is provided below.

The power generating cost functions for the power generators were given by:

$$f_1(q) = 2.5q_1^2 + q_1 q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1 q_2 + 2q_2, \quad f_3(q) = .5q_3^2 + .5q_1 q_3 + 2q_3.$$

The transaction cost functions faced by the power generators and associated with transacting with the retailers/suppliers were given by:

$$c_{11}(Q^1) = .5q_{11}^2 + 3.5q_{11},\ c_{12}(Q^1) = .5q_{12}^2 + 3.5q_{12},\ c_{21}(Q^1) = .5q_{21}^2 + 3.5q_{21},\ c_{22}(Q^1) = .5q_{22}^2 + 3.5q_{22},$$

$$c_{31}(Q^1) = .5q_{31}^2 + 2q_{31}, \quad c_{32}(Q^1) = .5q_{32}^2 + 2q_{32}.$$

The operating costs of the power generators, in turn, were given by:

$$c_1(Q^1, Q^2) = .5(\sum_{i=1}^{2} q_{i1})^2, \quad c_2(Q^1, Q^2) = .5(\sum_{i=1}^{2} q_{i2})^2.$$

The demand functions at the demand markets were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1100, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1100,$$

$$d_3(\rho_3) = -2\rho_{33} - 1.5\rho_{31} + 1200,$$

and the transaction costs between the competitive retailers and the consumers at the demand markets were given by:

$$\hat{c}_{11}^1(Q^2) = q_{11}^1 + 5, \quad \hat{c}_{12}^1(Q^2) = q_{12}^1 + 5, \quad \hat{c}_{13}^1(Q^2) = q_{13}^1 + 5,$$

$$\hat{c}_{21}^1(Q^2) = q_{21}^1 + 5, \quad \hat{c}_{22}^1(Q^2) = q_{22}^1 + 5, \quad \hat{c}_{23}^1(Q^2) = q_{23}^1 + 5.$$

All other transaction costs were assumed to be equal to zero.

We utilized the supernetwork representation of this example depicted in Figure 3 with the links enumerated as in Figure 3 in order to solve the problem via the Euler method. Note that there are 11 nodes and 17 links in the supernetwork in Figure 3. Using the procedure outlined in Section 4, we defined O/D pair $w_1 = (0, z_1)$, O/D pair $w_2 = (0, z_2)$, and O/D pair $w_3 = (0, z_3)$ and associated the demand price functions with the travel disutilities as in (56) and the user link travel cost functions as given in (50) – (53) (analogous constructions were done for the subsequent examples).

There were six paths in $P_{w_1}$ denoted by: $p_1, p_2, \ldots, p_6$; six paths in $P_{w_2}$ denoted, respectively, by $p_7, p_8, \ldots, p_{12}$, and also six paths in $P_{w_3}$ denoted, respectively, by $p_{13}, \ldots, p_{18}$. The paths were comprised of the following links: for O/D pair $w_1$:

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}), \quad p_3 = (a_2, a_{21}, a_{11'}, a_{1'1}),$$

$$p_4 = (a_2, a_{22}, a_{22'}, a_{2'1}), \quad p_5 = (a_3, a_{31}, a_{11'}, a_{1'1}), \quad p_6 = (a_3, a_{32}, a_{22'}, a_{2'1}),$$

for O/D pair $w_2$:

$$p_7 = (a_1, a_{11}, a_{11'}, a_{1'2}), \quad p_8 = (a_1, a_{12}, a_{22'}, a_{2'2}), \quad p_9 = (a_2, a_{21}, a_{11'}, a_{1'2}),$$

$$p_{10} = (a_2, a_{22}, a_{22'}, a_{2'2}), \quad p_{11} = (a_3, a_{31}, a_{11'}, a_{1'2}), \quad p_{12} = (a_3, a_{32}, a_{22'}, a_{2'2}),$$

for O/D pair $w_3$:

$$p_{13} = (a_1, a_{11}, a_{11'}, a_{1'3}), \quad p_{14} = (a_1, a_{12}, a_{22'}, a_{2'3}), \quad p_{15} = (a_2, a_{21}, a_{11'}, a_{1'3}),$$

$$p_{16} = (a_2, a_{22}, a_{22'}, a_{2'3}), \quad p_{17} = (a_3, a_{31}, a_{11'}, a_{1'3}), \quad p_{18} = (a_3, a_{32}, a_{22'}, a_{2'3}).$$

The Euler method converged in 374 iterations and yielded the following equilibrium path flow pattern:

$$x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = x^*_{p_4} = 4.758; \quad x^*_{p_5} = x^*_{p_6} = 10.871,$$

$$x^*_{p_7} = x^*_{p_8} = x^*_{p_9} = x^*_{p_{10}} = 4.759; \quad x^*_{p_{11}} = x^*_{p_{12}} = 10.867,$$

$$x^*_{p_{13}} = x^*_{p_{14}} = x^*_{p_{15}} = x^*_{p_{16}} = 4.759; \quad x^*_{p_{17}} = x^*_{p_{18}} = 35.867.$$

The computed equilibrium travel disutilities were:

$$\lambda^*_{w_1} = \lambda^*_{w_2} = 302.637; \quad \lambda^*_{w_3} = 327.637.$$

The corresponding equilibrium link flows (cf. also the supernetwork in Figure 3) were:

$$f_{a_1}^* = f_{a_2}^* = 28.553; \quad f_{a_3}^* = 115.211,$$

$$f_{a_{11}}^* = f_{a_{12}}^* = f_{a_{21}}^* = f_{a_{22}}^* = 14.277; \quad f_{a_{31}}^* = f_{a_{32}}^* = 57.605,$$

$$f_{a_{11'}}^* = f_{a_{22'}}^* = 86.158,$$

$$f_{a_{1'1}}^* = f_{a_{1'2}}^* = 20.386; \quad f_{a_{1'3}}^* = 45.386; \quad f_{a_{2'1}}^* = f_{a_{2'2}}^* = 20.386; \quad f_{a_{2'3}}^* = 45.386.$$

We now provide the translations of the above equilibrium flows into the electric power network flow and price notation using (40) – (43) and (45) – (49).

The power flows were:

$$q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 14.277; \quad q_{31}^* = q_{32}^* = 57.605,$$

$$q_{11}^{1*} = q_{12}^{1*} = q_{21}^{1*} = q_{22}^{1*} = 20.386; \quad q_{13}^{1*} = q_{23}^{1*} = 45.386.$$

The demand prices at the demand markets were:

$$\rho_{31}^* = \rho_{32}^* = 302.637; \quad \rho_{33}^* = 327.637.$$

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy.

These values are the same values obtained for this numerical example, but using the modified projection method by Nagurney and Matsypura (2004).

**Example 2**

We then solved the following variant of Example 1, which is also taken from Nagurney and Matsypura (2004). We kept the data identical to that in Example 1 except that we that we changed the first demand function so that:

$$d_1(\rho_3) = -2\rho_{33} - 1.5\rho_{31} + 1500.$$

The Euler method converged in 286 iterations and yielded the following equilibrium path flow pattern: for O/D pair $w_1$:

$$x_{p_1}^* = x_{p_2}^* = x_{p_3}^* = x_{p_4}^* = 19.600; \quad x_{p_5}^* = x_{p_6}^* = 78.900.$$

All other path flows were identically equal to 0.000.

The equilibrium travel disutilities were:

$$\lambda_{w_1}^* = 501.494; \quad \lambda_{w_2}^* = 173.879; \quad \lambda_{w_3}^* = 223.879.$$

The computed equilibrium link flows were now:

$$f_{a_1}^* = f_{a_2}^* = 39.200; \quad f_{a_3}^* = 157.798,$$

$$f_{a_{11}}^* = f_{a_{12}}^* = f_{a_{21}}^* = f_{a_{22}}^* = 19.600; \quad f_{a_{31}}^* = f_{a_{32}}^* = 78.899,$$

$$f_{a_{11'}}^* = f_{a_{22'}}^* = 118.099,$$

$$f_{a_{1'1}}^* = 118.099; \quad f_{a_{1'2}}^* = f_{a_{1'3}}^* = 0.000; \quad f_{a_{2'1}}^* = 118.099; \quad f_{a_{2'2}}^* = f_{a_{2'3}}^* = 0.000.$$

For easy reference and completeness, we now provide the translations of the above into the electric power notation using again (40) – (43) and (45) – (48). Specifically, we now had that:

$$q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 19.599; \quad q_{31}^* = q_{32}^* = 78.899,$$

$$q_{11}^{1*} = q_{21}^{1*} = 118.099,$$

and all other $q_{sk}^{1*}$s$= 0.0000$, and the demand prices at the demand markets were:

$$\rho_{31}^* = 501.494; \quad \rho_{32}^* = 173.879; \quad \rho_{33}^* = 223.879.$$

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy. These results are almost exactly those obtained for this example, but using the modified projection method by Nagurney and Matsypura (2004).

Note that with the increased demand at demand market 1 as evidenced through the new demand function, the demand price at that market increased. This was the only demand market that had positive electric power flowing into it; the other two demand markets had zero electric power consumed.

**Example 3**

Example 3 was constructed as follows from Example 2: The data were identical to that in Example 2 except that we changed the coefficient preceding the first term in the power generating function associated with the first power generator so that rather than having the term $2.5q_1^2$ in $f_1(q)$ there was now the term $5q_1^2$.

The Euler method converged in 313 iterations and yielded the following new equilibrium pattern: The computed equilibrium path flows were now:

$$x^*_{p_1} = x^*_{p_2} = 10.372; \quad x^*_{p_3} = x^*_{p_4} = 21.896; \quad x^*_{p_5} = x^*_{p_6} = 84.243.$$

All other path flows were, again, equal to 0.000 as in Example 2.

The computed equilibrium travel disutilities were:

$$\lambda^*_{w_1} = 505.122, \quad \lambda^*_{w_2} = 171.157, \quad \lambda^*_{w_3} = 221.157.$$

The computed equilibrium link flows were now:

$$f^*_{a_1} = 20.744; \quad f^*_{a_2} = 43.792; \quad f^*_{a_3} = 168.486,$$

$$f^*_{a_{11}} = f^*_{a_{12}} = 10.372; \quad f^*_{a_{21}} = f^*_{a_{22}} = 21.896; \quad f^*_{a_{31}} = f^*_{a_{32}} = 84.243,$$

$$f^*_{a_{11'}} = f^*_{a_{22'}} = 116.511,$$

$$f^*_{a_{1'1}} = 116.511; \quad f^*_{a_{1'2}} = f^*_{a_{1'3}} = 0.000; \quad f^*_{a_{2'1}} = 116.511; \quad f^*_{a_{2'2}} = f^*_{a_{2'3}} = 0.000.$$

The translation of these results using (40) – (43) and (45) – (49) corresponds to the following equilibrium solution in the elctric power network notation:

$$q^*_{11} = q^*_{12} = 10.372; \quad q^*_{21} = q^*_{22} = 21.896; \quad q^*_{31} = q^*_{32} = 84.243.$$

$$q^{1*}_{11} = q^{1*}_{21} = 116.511,$$

with all other $q_{sk}^{1*}$s$= 0.0000$.

The equilibrium demand prices at the demand markets were:

$$\rho_{31}^* = 505.122; \quad \rho_{32}^* = 171.157; \quad \rho_{33}^* = 221.157.$$

These results coincide with those obtained in Nagurney and Matsypura (2004). Also, as noted therein, since the power generating cost function associated with the first power generator increased, the power that he generated decreased; the power generated by the two other power generators, on the other hand, increased. Again, as in Example 2, there was no demand (at the computed equilibrium prices) at the second and third demand markets.

**Example 4**

The fourth, and final example, was constructed as follows from Example 3, and it is also taken from Nagurney and Matsypura (2004). The data were all as in Example 3, but now it was assumed that the demand functions were separable; hence, from each of the three demand market functions for electric power in Example 3, the term not corresponding to the price at the specific market was eliminated. In other words, the demand at demand market 1 only depended upon the price at demand market 1; the demand at demand market 2 only depended upon the demand at demand market 2; and the same held for the third demand market.

The Euler method (version 1) converged in 756 iterations and yielded the following equilibrium link flow pattern

$$f_{a_1}^* = 28.360; \quad f_{a_2}^* = 59.872; \quad f_{a_3}^* = 229.983,$$

$$f_{a_{11}}^* = f_{a_{12}}^* = 14.180; \quad f_{a_{21}}^* = f_{a_{22}}^* = 29.936; \quad f_{a_{31}}^* = f_{a_{32}}^* = 114.991,$$

$$f_{a_{11'}}^* = f_{a_{22'}}^* = 159.107,$$

$$f_{a_{1'1}}^* = 111.369; \quad f_{a_{1'2}}^* = 11.369; \quad f_{a_{1'3}}^* = 36.369,$$

$$f_{a_{2'1}}^* = 111.369; \quad f_{a_{2'2}}^* = 11.369; \quad f_{a_{2'3}}^* = 36.369.$$

which was identical to the equilibrium link flow pattern computed via the Euler method (version 2) but in 953 iterations.

The solution corresponds to the following equilibrium electric power flow and price pattern:

$$q_{11}^* = q_{12}^* = 14.1801; \quad q_{21}^* = q_{22}^* = 29.936; \quad q_{31}^* = q_{32}^* = 114.991,$$

$$q_{11}^{1*} = q_{21}^{1*} = 111.369; \quad q_{12}^{1*} = q_{22}^{1*} = 11.369; \quad q_{13}^{1*} = q_{23}^{1*} = 36.369.$$

The equilibrium demand prices at the demand markets were now:

$$\rho_{31}^* = 638.631; \quad \rho_{32}^* = 538.631; \quad \rho_{33}^* = 563.631.$$

The computed equilibrium link flows were almost precisely the same obtained by Nagurney and Matsypura (2004) who, however, only formulated and solved the electric power network example in the case of known demand functions.

Observe that since now there were no cross-terms in the demand functions, the electric power flows transacted between the competitive retailers and the demand markets were all positive. Of course, the incurred demands at both the second and third demand markets also increased. In addition, all the equilibrium flows from the power generators to the competitive retailers increased since there was increased demands at all the demand markets for electric power.

These numerical examples, although stylized, demonstrate the types of simulations that can be carried out. Indeed, one can easily investigate the effects on the equilibrium power flows and prices of such changes as: changes to the demand functions, to the power generating cost functions, as well as to the other cost functions. In addition, one can easily add or remove various decision-makers by changing the supply chain network structure (with the corresponding addition/removal of appropriate nodes and links) to investigate the effects of such market structure changes.

Also, we now have another interpretation of electric power network equilibrium as well as other algorithms that can be applied to solve either the case with known demand functions or the case with known inverse demand (or demand market price) functions.

## 6. Summary and Conclusions

In this paper, we have taken up the challenge, first posed in Beckmann, McGuire, and Winsten (1956) in Chapter 5 of their book on "some unsolved problems," of establishing the relationship and application of transportation network equilibrium models to electric power networks. As noted in that classic book on page 106, "The unsolved problems concern the application of this model to particular cases." "In particular, the problem of generation and distribution of electric energy in a network comes to mind."

Specifically, we utilized variational inequality theory to establish the equivalence between electric power supply chain network equilibria and transportation network equilibria with elastic demands over a specially-constructed supernetwork. We considered two formulations – with known demand functions and known demand market price (or disutility functions). The theoretical results established in this paper were then exploited in the computation of electric power numerical examples which were solved as reformulated transportation network equilibrium problems.

The results in this paper suggest several directions for future research, including, as noted earlier, the incorporation of complete electric power grids into the supernetwork, as well as the development of real-time dynamic versions of the electric power network models proposed in this paper. In particular, we note the potential of evolutionary variational inequalities and projected dynamical systems for this application domain (see, e.g., Cojocaru, Daniele, and Nagurney (2005a, b)).

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**Appendix** 1

Table 2: Definitions of the Feasible Sets for the Models

| **Notation** | **Definition** |
|---|---|
| $\mathcal{K}^1$ | $\{(q, Q^1) \vert (q, Q^1) \in R_+^{G+GS}$ and (1) holds$\}$ |
| $\mathcal{K}^2$ | $\{(Q^2, Q^1) \vert (Q^2, Q^1) \in R_+^{SKV+GS}$ and (8) holds$\}$ |
| $\mathcal{K}^3$ | $\{(h, Q^2, Q^1) \vert (h, Q^2, Q^1) \in R_+^{S(1+VK+G)}$ and (8) and (13) hold$\}$ |
| $\mathcal{K}^4$ | $\{(Q^2, d, \rho_3) \vert (Q^2, d, \rho_3) \in R_+^{K(SV+2)}$ and (19) holds$\}$ |
| $\mathcal{K}^5$ | $\{(Q^2, d) \vert (Q^2, d) \in R_+^{K(SV+1)}$ and (19) holds$\}$ |
| $\mathcal{K}^6$ | $\{(q, h, Q^1, Q^2, d, \rho_3) \vert (q, h, Q^1, Q^2, d, \rho_3) \in R_+^{G+S+GS+VSK+2K}$ and (1), (8), (13), *and* (19) hold$\}$ |
| $\mathcal{K}^7$ | $\{(q, h, Q^1, Q^2, d) \vert (q, h, Q^1, Q^2, d) \in R_+^{G+S+GS+VSK+K}$ and (1), (8), (13), *and* (19) hold.$\}$ |
| $\mathcal{K}^8$ | $\{(f, d, \lambda) \in R_+^{K+2Z} \vert$ there exists an $x$ satisfying (27) and $d_w = \sum_{p \in P_w} x_p, \forall w\}$ |
| $\mathcal{K}^9$ | $\{(x, d) \vert (x, d) \in R_+^{K+Z}$ and $d_w = \sum_{p \in P_w} x_p, \forall w\}$ |
| $\mathcal{K}^{10}$ | $\{(f, d) \in R_+^{K+Z} \vert$ there exists an $x$ satisfying (27) and $d_w = \sum_{p \in P_w} x_p, \forall w\}$ |

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